CHAPTER 3
INNER PRODUCT SPACES. HILBERT SPACES

In a normed space we can add vectors and multiply vectors by scalars, just as in elementary vector algebra. Furthermore, the norm on such a space generalizes the elementary concept of the length of a vector. However, what is still missing in a general normed space, and what we would like to have if possible, is an analogue of the familiar dot product

$$a \cdot b = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

and resulting formulas, notably

$$|a| = \sqrt{a \cdot a}$$

and the condition for orthogonality (perpendicularity)

$$a \cdot b = 0$$

which are important tools in many applications. Hence the question arises whether the dot product and orthogonality can be generalized to arbitrary vector spaces. In fact, this can be done and leads to *inner product spaces* and complete inner product spaces, called *Hilbert spaces.*

Inner product spaces are special normed spaces, as we shall see. Historically they are older than general normed spaces. Their theory is richer and retains many features of Euclidean space, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean space, and the reader should note the great harmony and beauty of the concepts and proofs in this field. The whole theory was initiated by the work of D. Hilbert (1912) on integral equations. The currently used geometrical notation and terminology is analogous to that of Euclidean geometry and was coined by E. Schmidt (1908), who followed a suggestion of G. Kowalewski (as he mentioned on p. 56 of his paper). These spaces have been, up to now, the most useful spaces in practical applications of functional analysis.

Important concepts, brief orientation about main content

An *inner product space* X (Def. 3.1-1) is a vector space with an *inner product* $\langle x, y\rangle$ defined on it. The latter generalizes the dot product of vectors in three dimensional space and is used to define

(I) a *norm* $\|\cdot\|$ by $\|x\|=\langle x, x\rangle^{1/2}$,

(II) *orthogonality* by $\langle x, y\rangle=0$.

A *Hilbert space* H is a complete inner product space. The theory of inner product and Hilbert spaces is richer than that of general normed and Banach spaces. Distinguishing features are

(i) representations of H as a direct sum of a closed subspace and its *orthogonal complement* (cf. 3.3-4),

(ii) *orthonormal sets and sequences* and corresponding representations of elements of H (cf. Secs. 3.4, 3.5),

(iii) the *Riesz representation* 3.8-1 of bounded linear functionals by inner products,

(iv) the *Hilbert-adjoint operator* T^* of a bounded linear operator T (cf. 3.9-1).

Orthonormal sets and sequences are truly interesting only if they are total (Sec. 3.6). Hilbert-adjoint operators can be used to define classes of operators (*self-adjoint, unitary, normal;* cf. Sec. 3.10) which are of great importance in applications.

3.1 Inner Product Space. Hilbert Space

The spaces to be considered in this chapter are defined as follows.

3.1-1 Definition (Inner product space, Hilbert space). An *inner product space* (or *pre-Hilbert space*) is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space (complete in the metric defined by the inner product; cf. (2), below). Here, an **inner product** *on* X is a mapping of $X\times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, y\rangle$$

and is called the *inner product*[1] of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(IP2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(IP3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(IP4) $\langle x, x\rangle \geqq 0$

$\langle x, x\rangle = 0 \iff x = 0.$

An inner product on X defines a *norm* on X given by

$$\|x\| = \sqrt{\langle x, x\rangle} \qquad (\geqq 0) \tag{1}$$

and a *metric* on X given by

$$d(x, y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}. \tag{2}$$ ∎

Hence *inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.*

In (IP3), the bar denotes complex conjugation. Consequently, if X is a *real* vector space, we simply have

$$\langle x, y\rangle = \langle y, x\rangle \qquad (\textit{Symmetry}).$$

The proof that (1) satisfies the axioms (N1) to (N4) of a norm (cf. Sec. 2.2) will be given at the beginning of the next section.

From (IP1) to (IP3) we obtain the formula

(3)

(a) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

(b) $\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle$

(c) $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$

[1] Or *scalar product*, but this must not be confused with the product of a vector by a scalar in a vector space.

The notation $\langle\ ,\ \rangle$ for the inner product is quite common. In an elementary text such as the present one it may have the advantage over another popular notation, $(\ ,\ )$, that it excludes confusion with ordered pairs (components of a vector, elements of a product space, arguments of functions depending on two variables, etc.).

which we shall use quite often. (3a) shows that the inner product is linear in the first factor. Since in (3c) we have complex conjugates $\bar{\alpha}$ and $\bar{\beta}$ on the right, we say that the inner product is *conjugate linear* in the second factor. Expressing both properties together, we say that the inner product is *sesquilinear.* This means "$1\frac{1}{2}$ times linear" and is motivated by the fact that "conjugate linear" is also known as "semilinear" (meaning "halflinear"), a less suggestive term which we shall not use.

The reader may show by a simple straightforward calculation that a norm on an inner product space satisfies the important **parallelogram equality**

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2). \tag{4}$$

This name is suggested by elementary geometry, as we see from Fig. 23 if we remember that the norm generalizes the elementary concept of the length of a vector (cf. Sec. 2.2). It is quite remarkable that such an equation continues to hold in our present much more general setting.

We conclude that if a norm does not satisfy (4), it cannot be obtained from an inner product by the use of (1). Such norms do exist; examples will be given below. Without risking misunderstandings we may thus say:

Not all normed spaces are inner product spaces.

Before we consider examples, let us define the concept of orthogonality, which is basic in the whole theory. We know that if the dot product of two vectors in three dimensional spaces is zero, the vectors are orthogonal, that is, they are perpendicular or at least one of them is the zero vector. This suggests and motivates the following

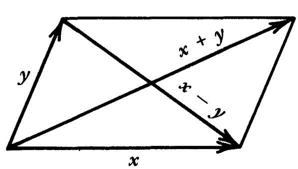

Fig. 23. Parallelogram with sides x and y in the plane

3.1-2 Definition (Orthogonality). An element x of an inner product space X is said to be *orthogonal* to an element $y \in X$ if

$$\langle x, y \rangle = 0.$$

We also say that *x and y are orthogonal*, and we write $x \perp y$. Similarly, for subsets A, $B \subset X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$. ▮

Examples

3.1-3 Euclidean space $\mathbf{R}^n$. The space $\mathbf{R}^n$ is a Hilbert space with inner product defined by

$$\langle x, y \rangle = \xi_1 \eta_1 + \cdots + \xi_n \eta_n \tag{5}$$

where $x = (\xi_j) = (\xi_1, \cdots, \xi_n)$ and $y = (\eta_j) = (\eta_1, \cdots, \eta_n)$.

In fact, from (5) we obtain

$$\|x\| = \langle x, x \rangle^{1/2} = (\xi_1{}^2 + \cdots + \xi_n)^{1/2}$$

and from this the Euclidean metric defined by

$$d(x, y) = \|x - y\| = \langle x - y, x - y \rangle^{1/2} = [(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2]^{1/2};$$

cf. 2.2-2. Completeness was shown in 1.5-1.

If $n = 3$, formula (5) gives the usual dot product

$$\langle x, y \rangle = x \cdot y = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3$$

of $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$, and the orthogonality

$$\langle x, y \rangle = x \cdot y = 0$$

agrees with the elementary concept of perpendicularity.

3.1-4 Unitary space $\mathbf{C}^n$. The space $\mathbf{C}^n$ defined in 2.2-2 is a Hilbert space with inner product given by

$$\langle x, y \rangle = \xi_1 \bar{\eta}_1 + \cdots + \xi_n \bar{\eta}_n. \tag{6}$$

In fact, from (6) we obtain the norm defined by

$$\|x\| = (\xi_1\bar{\xi}_1 + \cdots + \xi_n\bar{\xi}_n)^{1/2} = (|\xi_1|^2 + \cdots + |\xi_n|^2)^{1/2}.$$

Here we also see why we have to take complex conjugates $\bar{\eta}_j$ in (6); this entails $\langle y, x\rangle = \overline{\langle x, y\rangle}$, which is (IP3), so that $\langle x, x\rangle$ is real.

3.1-5 Space $L^2[a, b]$. The norm in Example 2.2-7 is defined by

$$\|x\| = \left(\int_a^b x(t)^2 \, dt\right)^{1/2}$$

and can be obtained from the inner product defined by

$$\langle x, y\rangle = \int_a^b x(t)y(t) \, dt. \tag{7}$$

In Example 2.2-7 the functions were assumed to be real-valued, for simplicity. In connection with certain applications it is advantageous to remove that restriction and consider *complex-valued* functions (keeping $t \in [a, b]$ real, as before). These functions form a complex vector space, which becomes an inner product space if we define

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)} \, dt. \tag{7*}$$

Here the bar denotes the complex conjugate. It has the effect that (IP3) holds, so that $\langle x, x\rangle$ is still real. This property is again needed in connection with the norm, which is now defined by

$$\|x\| = \left(\int_a^b |x(t)|^2 \, dt\right)^{1/2}$$

because $x(t)\overline{x(t)} = |x(t)|^2$.

The completion of the metric space corresponding to (7) is the real space $L^2[a, b]$; cf. 2.2-7. Similarly, the completion of the metric space corresponding to (7*) is called the *complex space* $L^2[a, b]$. We shall see in the next section that the inner product can be extended from an inner product space to its completion. Together with our present discussion this implies that $L^2[a, b]$ is a Hilbert space.

3.1-6 Hilbert sequence space l^2. The space l^2 (cf. 2.2-3) is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \bar{\eta}_j. \tag{8}$$

Convergence of this series follows from the Cauchy-Schwarz inequality (11), Sec. 1.2, and the fact that $x, y \in l^2$, by assumption. We see that (8) generalizes (6). The norm is defined by

$$\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{1/2}.$$

Completeness was shown in 1.5-4.

l^2 is the prototype of a Hilbert space. It was introduced and investigated by D. Hilbert (1912) in his work on integral equations. An axiomatic definition of Hilbert space was not given until much later, by J. von Neumann (1927), pp. 15–17, in a paper on the mathematical foundation of quantum mechanics. Cf. also J. von Neumann (1929–30), pp. 63–66, and M. H. Stone (1932), pp. 3–4. That definition included separability, a condition which was later dropped from the definition when H. Löwig (1934), F. Rellich (1934) and F. Riesz (1934) showed that for most parts of the theory that condition was an unnecessary restriction. (These papers are listed in Appendix 3.)

3.1-7 Space l^p. *The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.*

Proof. Our statement means that the norm of l^p with $p \neq 2$ cannot be obtained from an inner product. We prove this by showing that the norm does not satisfy the parallelogram equality (4). In fact, let us take $x = (1, 1, 0, 0, \cdots) \in l^p$ and $y = (1, -1, 0, 0, \cdots) \in l^p$ and calculate

$$\|x\| = \|y\| = 2^{1/p}, \qquad \|x+y\| = \|x-y\| = 2.$$

We now see that (4) is not satisfied if $p \neq 2$.

l^p is complete (cf. 1.5-4). Hence *l^p with $p \neq 2$ is a Banach space which is not a Hilbert space.* The same holds for the space in the next example.

3.1-8 Space $C[a, b]$. *The space $C[a, b]$ is not an inner product space, hence not a Hilbert space.*

Proof. We show that the norm defined by

$$\|x\| = \max_{t \in J} |x(t)| \qquad J = [a, b]$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram equality (4). Indeed, if we take $x(t) = 1$ and $y(t) = (t-a)/(b-a)$, we have $\|x\| = 1$, $\|y\| = 1$ and

$$x(t) + y(t) = 1 + \frac{t-a}{b-a}$$

$$x(t) - y(t) = 1 - \frac{t-a}{b-a}.$$

Hence $\|x+y\| = 2$, $\|x-y\| = 1$ and

$$\|x+y\|^2 + \|x-y\|^2 = 5 \qquad \text{but} \qquad 2(\|x\|^2 + \|y\|^2) = 4.$$

This completes the proof. ∎

We finally mention the following interesting fact. We know that to an inner product there corresponds a norm which is given by (1). It is remarkable that, conversely, we can "rediscover" the inner product from the corresponding norm. In fact, the reader may verify by straightforward calculation that for a real inner product space we have

(9) $$\langle x, y \rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

and for a complex inner product space we have

(10) $$\begin{aligned} \operatorname{Re} \langle x, y \rangle &= \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ \operatorname{Im} \langle x, y \rangle &= \tfrac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2). \end{aligned}$$

Formula (10) is sometimes called the **polarization identity.**

Problems

1. Prove (4).

2. (Pythagorean theorem) If $x \perp y$ in an inner product space X, show that (Fig. 24)

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Extend the formula to m mutually orthogonal vectors.

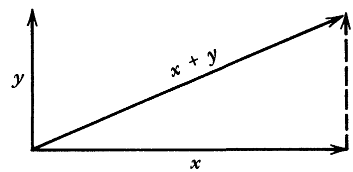

Fig. 24. Illustration of the Pythagorean theorem in the plane

3. If X in Prob. 2 is real, show that, conversely, the given relation implies that $x \perp y$. Show that this may not hold if X is complex. Give examples.

4. If an inner product space X is real, show that the condition $\|x\| = \|y\|$ implies $\langle x+y, x-y\rangle = 0$. What does this mean geometrically if $X = \mathbf{R}^2$? What does the condition imply if X is complex?

5. (Appolonius' identity) Verify by direct calculation that for any elements in an inner product space,

$$\|z-x\|^2 + \|z-y\|^2 = \tfrac{1}{2}\|x-y\|^2 + 2\,\|z - \tfrac{1}{2}(x+y)\|^2.$$

Show that this identity can also be obtained from the parallelogram equality.

6. Let $x \neq 0$ and $y \neq 0$. (*a*) If $x \perp y$, show that $\{x, y\}$ is a linearly independent set. (*b*) Extend the result to mutually orthogonal nonzero vectors $x_1, \cdots, x_m$.

7. If in an inner product space, $\langle x, u\rangle = \langle x, v\rangle$ for all x, show that $u = v$.

8. Prove (9).

9. Prove (10).

10. Let z_1 and z_2 denote complex numbers. Show that $\langle z_1, z_2\rangle = z_1\bar{z}_2$ defines an inner product, which yields the usual metric on the complex plane. Under what condition do we have orthogonality?

11. Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm defined on X by

$$\|x\| = |\xi_1| + |\xi_2| \qquad [x = (\xi_1, \xi_2)]$$

from an inner product?

12. What is $\|x\|$ in 3.1-6 if $x = (\xi_1, \xi_2, \cdots)$, where (*a*) $\xi_n = 2^{-n/2}$, (*b*) $\xi_n = 1/n$?

13. Verify that for continuous functions the inner product in 3.1-5 satisfies (IP1) to (IP4).

14. Show that the norm on $C[a, b]$ is invariant under a linear transformation $t = \alpha\tau + \beta$. Use this to prove the statement in 3.1-8 by mapping $[a, b]$ onto $[0, 1]$ and then considering the functions defined by $\tilde{x}(\tau) = 1$, $\tilde{y}(\tau) = \tau$, where $\tau \in [0, 1]$.

15. If X is a finite dimensional vector space and (e_j) is a basis for X, show that an inner product on X is completely determined by its values $\gamma_{jk} = \langle e_j, e_k\rangle$. Can we choose such scalars γ_{jk} in a completely arbitrary fashion?

3.2 Further Properties of Inner Product Spaces

First of all, we should verify that (1) in the preceding section defines a norm:

(N1) and (N2) in Sec. 2.2 follow from (IP4). Furthermore, (N3) is obtained by the use of (IP2) and (IP3); in fact,

$$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\bar{\alpha}\langle x, x\rangle = |\alpha|^2 \|x\|^2.$$

Finally, (N4) is included in

3.2-1 Lemma (Schwarz inequality, triangle inequality). *An inner product and the corresponding norm satisfy the Schwarz inequality and the triangle inequality as follows.*

(a) *We have*

(1) $$|\langle x, y\rangle| \leqq \|x\| \, \|y\| \qquad \textbf{(Schwarz inequality)}$$

where the equality sign holds if and only if $\{x, y\}$ *is a linearly dependent set.*

(b) *That norm also satisfies*

(2) $$\|x+y\| \leqq \|x\| + \|y\| \qquad \textbf{(Triangle inequality)}$$

where the equality sign holds if and only if[2] $y=0$ *or* $x=cy$ (c *real and* $\geqq 0$).

Proof. **(a)** If $y=0$, then (1) holds since $\langle x, 0\rangle = 0$. Let $y \neq 0$. For every scalar α we have

$$\begin{aligned} 0 \leqq \|x-\alpha y\|^2 &= \langle x-\alpha y, x-\alpha y\rangle \\ &= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha[\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle]. \end{aligned}$$

We see that the expression in the brackets $[\cdots]$ is zero if we choose $\bar{\alpha} = \langle y, x\rangle / \langle y, y\rangle$. The remaining inequality is

$$0 \leqq \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle = \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2};$$

here we used $\langle y, x\rangle = \overline{\langle x, y\rangle}$. Multiplying by $\|y\|^2$, transferring the last term to the left and taking square roots, we obtain (1).

Equality holds in this derivation if and only if $y=0$ or $0 = \|x-\alpha y\|^2$, hence $x - \alpha y = 0$, so that $x = \alpha y$, which shows linear dependence.

(b) We prove (2). We have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2.$$

By the Schwarz inequality,

$$|\langle x, y\rangle| = |\langle y, x\rangle| \leqq \|x\| \, \|y\|.$$

[2] Note that this condition for equality is perfectly "symmetric" in x and y since $x=0$ is included in $x=cy$ (for $c=0$) and so is $y=kx$, $k=1/c$ (for $c>0$).

By the triangle inequality for numbers we thus obtain

$$\begin{aligned}\|x+y\|^2 &\leqq \|x\|^2+2\,|\langle x, y\rangle|+\|y\|^2 \\ &\leqq \|x\|^2+2\,\|x\|\,\|y\|+\|y\|^2 \\ &= (\|x\|+\|y\|)^2.\end{aligned}$$

Taking square roots on both sides, we have (2).

Equality holds in this derivation if and only if

$$\langle x, y\rangle+\langle y, x\rangle=2\,\|x\|\,\|y\|.$$

The left-hand side is $2\operatorname{Re}\langle x, y\rangle$, where Re denotes the real part. From this and (1),

$$\operatorname{Re}\langle x, y\rangle=\|x\|\,\|y\|\geqq|\langle x, y\rangle|. \tag{3}$$

Since the real part of a complex number cannot exceed the absolute value, we must have equality, which implies linear dependence by part (a), say, $y=0$ or $x=cy$. We show that c is real and $\geqq 0$. From (3) with the equality sign we have $\operatorname{Re}\langle x, y\rangle=|\langle x, y\rangle|$. But if the real part of a complex number equals the absolute value, the imaginary part must be zero. Hence $\langle x, y\rangle=\operatorname{Re}\langle x, y\rangle\geqq 0$ by (3), and $c\geqq 0$ follows from

$$0\leqq\langle x, y\rangle=\langle cy, y\rangle=c\,\|y\|^2. \qquad \blacksquare$$

The Schwarz inequality (1) is quite important and will be used in proofs over and over again. Another frequently used property is the continuity of the inner product:

3.2-2 Lemma (Continuity of inner product). *If in an inner product space, $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle$.*

Proof. Subtracting and adding a term, using the triangle inequality for numbers and, finally, the Schwarz inequality, we obtain

$$\begin{aligned}|\langle x_n, y_n\rangle-\langle x, y\rangle| &= |\langle x_n, y_n\rangle-\langle x_n, y\rangle+\langle x_n, y\rangle-\langle x, y\rangle| \\ &\leqq |\langle x_n, y_n-y\rangle|+|\langle x_n-x, y\rangle| \\ &\leqq \|x_n\|\,\|y_n-y\|+\|x_n-x\|\,\|y\| \quad\longrightarrow\quad 0\end{aligned}$$

since $y_n-y \longrightarrow 0$ and $x_n-x \longrightarrow 0$ as $n \longrightarrow \infty$. $\blacksquare$

As a first application of this lemma, let us prove that every inner product space can be completed. The completion is a Hilbert space and is unique except for isomorphisms. Here the definition of an isomorphism is as follows (as suggested by our discussion in Sec. 2.8).

An **isomorphism** T of an inner product space X onto an inner product space $\tilde{X}$ over the same field is a bijective linear operator $T\colon X \longrightarrow \tilde{X}$ which preserves the inner product, that is, for all $x, y \in X$,

$$\langle Tx, Ty \rangle = \langle x, y \rangle,$$

where we denoted inner products on X and $\tilde{X}$ by the same symbol, for simplicity. $\tilde{X}$ is then called *isomorphic* with X, and X and $\tilde{X}$ are called *isomorphic inner product spaces.* Note that the bijectivity and linearity guarantees that T is a vector space isomorphism of X onto $\tilde{X}$, so that T preserves the whole structure of inner product space. T is also an isometry of X onto $\tilde{X}$ because distances in X and $\tilde{X}$ are determined by the norms defined by the inner products on X and $\tilde{X}$.

The theorem about the completion of an inner product space can now be stated as follows.

3.2-3 Theorem (Completion). *For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $W \subset H$. The space H is unique except for isomorphisms.*

Proof. By Theorem 2.3-2 there exists a Banach space H and an isometry A from X onto a subspace W of H which is dense in H. For reasons of continuity, under such an isometry, sums and scalar multiples of elements in X and W correspond to each other, so that A is even an isomorphism of X onto W, both regarded as normed spaces. Lemma 3.2-2 shows that we can define an inner product on H by setting

$$\langle \hat{x}, \hat{y} \rangle = \lim_{n \to \infty} \langle x_n, y_n \rangle,$$

the notations being as in Theorem 2.3-2 (and 1.6-2), that is, (x_n) and (y_n) are representatives of $\hat{x} \in H$ and $\hat{y} \in H$, respectively. Taking (9) and (10), Sec. 3.1, into account, we see that A is an isomorphism of X onto W, both regarded as inner product spaces.

Theorem 2.3-2 also guarantees that H is unique except for isometries, that is, two completions H and $\tilde{H}$ of X are related by an isometry $T\colon H \longrightarrow \tilde{H}$. Reasoning as in the case of A, we conclude that T must be an isomorphism of the Hilbert space H onto the Hilbert space $\tilde{H}$. ∎

A **subspace** Y of an inner product space X is defined to be a vector subspace of X (cf. Sec. 2.1) taken with the inner product on X restricted to $Y \times Y$.

Similarly, a **subspace** Y of a Hilbert space H is defined to be a subspace of H, regarded as an inner product space. Note that Y need not be a Hilbert space because Y may not be complete. In fact, from Theorems 2.3-1 and 2.4-2 we immediately have the statements (a) and (b) in the following theorem.

3.2-4 Theorem (Subspace). *Let Y be a subspace of a Hilbert space H. Then:*

(a) *Y is complete if and only if Y is closed in H.*

(b) *If Y is finite dimensional, then Y is complete.*

(c) *If H is separable, so is Y. More generally, every subset of a separable inner product space is separable.*

The simple proof of (c) is left to the reader.

Problems

1. What is the Schwarz inequality in $\mathbf{R}^2$ or $\mathbf{R}^3$? Give another proof of it in these cases.

2. Give examples of subspaces of l^2.

3. Let X be the inner product space consisting of the polynomial $x = 0$ (cf. the remark in Prob. 9, Sec. 2.9) and all real polynomials in t, of degree not exceeding 2, considered for real $t \in [a, b]$, with inner product defined by (7), Sec. 3.1. Show that X is complete. Let Y consist of all $x \in X$ such that $x(a) = 0$. Is Y a subspace of X? Do all $x \in X$ of degree 2 form a subspace of X?

4. Show that $y \perp x_n$ and $x_n \longrightarrow x$ together imply $x \perp y$.

5. Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \longrightarrow \|x\|$ and $\langle x_n, x\rangle \longrightarrow \langle x, x\rangle$ imply convergence $x_n \longrightarrow x$.

6. Prove the statement in Prob. 5 for the special case of the complex plane.

7. Show that in an inner product space, $x \perp y$ if and only if we have $\|x+\alpha y\| = \|x-\alpha y\|$ for all scalars α. (See Fig. 25.)

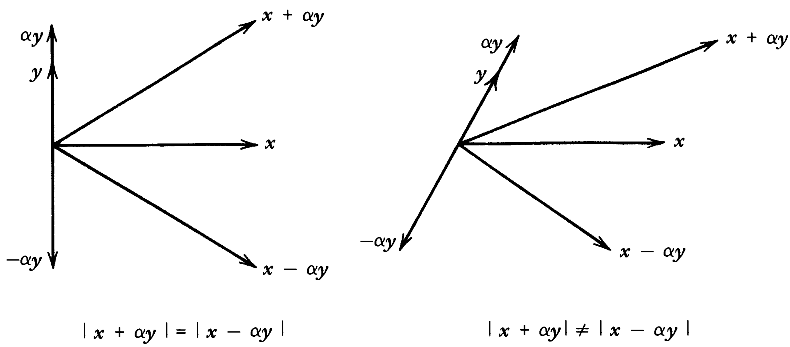

Fig. 25. Illustration of Prob. 7 in the Euclidean plane $\mathbf{R}^2$

8. Show that in an inner product space, $x \perp y$ if and only if $\|x+\alpha y\| \geqq \|x\|$ for all scalars α.

9. Let V be the vector space of all continuous complex-valued functions on $J=[a, b]$. Let $X_1 = (V, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{t \in J} |x(t)|$; and let $X_2 = (V, \|\cdot\|_2)$, where

$$\|x\|_2 = \langle x, x\rangle^{1/2}, \qquad \langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\, dt.$$

Show that the identity mapping $x \longmapsto x$ of X_1 onto X_2 is continuous. (It is not a homeomorphism. X_2 is not complete.)

10. (Zero operator) Let $T\colon X \longrightarrow X$ be a bounded linear operator on a complex inner product space X. *If* $\langle Tx, x\rangle = 0$ for all $x \in X$, show that $T = 0$.

Show that this does not hold in the case of a *real* inner product space. *Hint.* Consider a rotation of the Euclidean plane.

3.3 Orthogonal Complements and Direct Sums

In a metric space X, the *distance* δ from an element $x \in X$ to a nonempty subset $M \subset X$ is defined to be

$$\delta = \inf_{\tilde{y} \in M} d(x, \tilde{y}) \qquad (M \neq \varnothing).$$

In a normed space this becomes

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| \qquad (M \neq \varnothing). \tag{1}$$

A simple illustrative example is shown in Fig. 26.

We shall see that it is important to know whether there is a $y \in M$ such that

$$\delta = \|x - y\|, \tag{2}$$

that is, intuitively speaking, a point $y \in M$ which is closest to the given x, and if such an element exists, whether it is unique. This is an *existence and uniqueness problem.* It is of fundamental importance, theoretically as well as in applications, for instance, in connection with approximations of functions.

Figure 27 illustrates that even in a very simple space such as the Euclidean plane $\mathbf{R}^2$, there may be no y satisfying (2), or precisely one such y, or more than one y. And we may expect that other spaces, in particular infinite dimensional ones, will be much more complicated in that respect. For general normed spaces this is the case (as we shall see in Chap. 6), but for Hilbert spaces the situation remains relatively

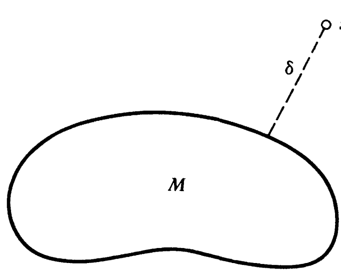

Fig. 26. Illustration of (1) in the case of the plane $\mathbf{R}^2$

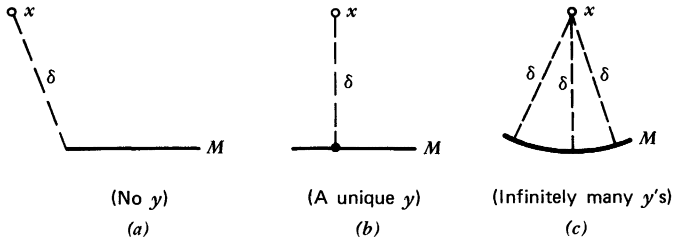

Fig. 27. Existence and uniqueness of points $y \in M$ satisfying (2), where the given $M \subset \mathbf{R}^2$ is an open segment [in (a) and (b)] and a circular arc [in (c)]

simple. This fact is surprising and has various theoretical and practical consequences. It is one of the main reasons why the theory of Hilbert spaces is simpler than that of general Banach spaces.

To consider that existence and uniqueness problem for Hilbert spaces and to formulate the key theorem (3.3-1, below), we need two related concepts, which are of general interest, as follows.

The **segment** joining two given elements x and y of a vector space X is defined to be the set of all $z \in X$ of the form

$$z = \alpha x + (1-\alpha) y \qquad (\alpha \in \mathbf{R}, 0 \leqq \alpha \leqq 1).$$

A subset M of X is said to be **convex** if for every $x, y \in M$ the segment joining x and y is contained in M. Figure 28 shows a simple example.

For instance, every subspace Y of X is convex, and the intersection of convex sets is a convex set.

We can now provide the main tool in this section:

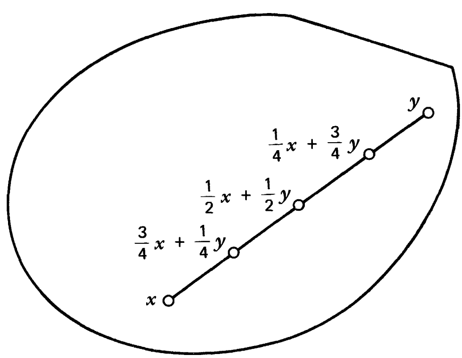

Fig. 28. Illustrative example of a segment in a convex set

3.3-1 Theorem (Minimizing vector). *Let X be an inner product space and $M \neq \varnothing$ a convex subset which is complete (in the metric induced by the inner product). Then for every given $x \in X$ there exists a unique $y \in M$ such that*

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|. \tag{3}$$

Proof. **(a)** *Existence.* By the definition of an infimum there is a sequence (y_n) in M such that

$$\delta_n \longrightarrow \delta \qquad \text{where} \qquad \delta_n = \|x - y_n\|. \tag{4}$$

We show that (y_n) is Cauchy. Writing $y_n - x = v_n$, we have $\|v_n\| = \delta_n$ and

$$\|v_n + v_m\| = \|y_n + y_m - 2x\| = 2\, \|\tfrac{1}{2}(y_n + y_m) - x\| \geqq 2\delta$$

because M is convex, so that $\frac{1}{2}(y_n + y_m) \in M$. Furthermore, we have $y_n - y_m = v_n - v_m$. Hence by the parallelogram equality,

$$\begin{aligned}\|y_n - y_m\|^2 = \|v_n - v_m\|^2 &= -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)\\ &\leqq -(2\delta)^2 + 2(\delta_n^{\ 2} + \delta_m^{\ 2}),\end{aligned}$$

and (4) implies that (y_n) is Cauchy. Since M is complete, (y_n) converges, say, $y_n \longrightarrow y \in M$. Since $y \in M$, we have $\|x - y\| \geqq \delta$. Also, by (4),

$$\|x - y\| \leqq \|x - y_n\| + \|y_n - y\| = \delta_n + \|y_n - y\| \quad \longrightarrow \quad \delta.$$

This shows that $\|x - y\| = \delta$.

(b) *Uniqueness.* We assume that $y \in M$ and $y_0 \in M$ both satisfy

$$\|x - y\| = \delta \qquad \text{and} \qquad \|x - y_0\| = \delta$$

and show that then $y_0 = y$. By the parallelogram equality,

$$\begin{aligned}\|y - y_0\|^2 &= \|(y - x) - (y_0 - x)\|^2\\ &= 2\,\|y - x\|^2 + 2\,\|y_0 - x\|^2 - \|(y - x) + (y_0 - x)\|^2\\ &= 2\delta^2 + 2\delta^2 - 2^2\,\|\tfrac{1}{2}(y + y_0) - x\|^2.\end{aligned}$$

On the right, $\frac{1}{2}(y+y_0)\in M$, so that

$$\|\tfrac{1}{2}(y+y_0)-x\|\geqq\delta.$$

This implies that the right-hand side is less than or equal to $2\delta^2+2\delta^2-4\delta^2=0$. Hence we have the inequality $\|y-y_0\|\leqq 0$. Clearly, $\|y-y_0\|\geqq 0$, so that we must have equality, and $y_0=y$. ∎

Turning from arbitrary convex sets to subspaces, we obtain a lemma which generalizes the familiar idea of elementary geometry that the unique point y in a given subspace Y closest to a given x is found by "dropping a perpendicular from x to Y."

3.3-2 Lemma (Orthogonality). *In Theorem* 3.3-1, *let M be a complete subspace Y and $x\in X$ fixed. Then $z=x-y$ is orthogonal to Y.*

Proof. If $z\perp Y$ were false, there would be a $y_1\in Y$ such that

$$\langle z, y_1\rangle=\beta\neq 0. \tag{5}$$

Clearly, $y_1\neq 0$ since otherwise $\langle z, y_1\rangle=0$. Furthermore, for any scalar α,

$$\begin{aligned}\|z-\alpha y_1\|^2&=\langle z-\alpha y_1, z-\alpha y_1\rangle\\&=\langle z, z\rangle-\bar{\alpha}\langle z, y_1\rangle-\alpha[\langle y_1, z\rangle-\bar{\alpha}\langle y_1, y_1\rangle]\\&=\langle z, z\rangle-\bar{\alpha}\beta-\alpha[\bar{\beta}-\bar{\alpha}\langle y_1, y_1\rangle].\end{aligned}$$

The expression in the brackets $[\cdots]$ is zero if we choose

$$\bar{\alpha}=\frac{\bar{\beta}}{\langle y_1, y_1\rangle}.$$

From (3) we have $\|z\|=\|x-y\|=\delta$, so that our equation now yields

$$\|z-\alpha y_1\|^2=\|z\|^2-\frac{|\beta|^2}{\langle y_1, y_1\rangle}<\delta^2.$$

But this is impossible because we have

$$z-\alpha y_1=x-y_2 \qquad \text{where} \qquad y_2=y+\alpha y_1\in Y,$$

so that $\|z-\alpha y_1\|\geqq\delta$ by the definition of δ. Hence (5) cannot hold, and the lemma is proved. ∎

Our goal is a representation of a Hilbert space as a direct sum which is particularly simple and suitable because it makes use of orthogonality. To understand the situation and the problem, let us first introduce the concept of a direct sum. This concept makes sense for any vector space and is defined as follows.

3.3-3 Definition (Direct sum). A vector space X is said to be the *direct sum* of two subspaces Y and Z of X, written

$$X = Y \oplus Z,$$

if each $x \in X$ has a unique representation

$$x = y + z \qquad\qquad y \in Y,\ z \in Z.$$

Then Z is called an *algebraic complement* of Y in X and vice versa, and Y, Z is called a *complementary pair* of subspaces in X. ∎

For example, $Y = \mathbf{R}$ is a subspace of the Euclidean plane $\mathbf{R}^2$. Clearly, Y has infinitely many algebraic complements in $\mathbf{R}^2$, each of which is a real line. But most convenient is a complement that is perpendicular. We make use of this fact when we choose a Cartesian coordinate system. In $\mathbf{R}^3$ the situation is the same in principle.

Similarly, in the case of a general Hilbert space H, the main interest concerns representations of H as a direct sum of a closed subspace Y and its **orthogonal complement**

$$Y^\perp = \{z \in H \mid z \perp Y\},$$

which is the set of all vectors orthogonal to Y. This gives our main result in this section, which is sometimes called the *projection theorem*, for reasons to be explained after the proof.

3.3-4 Theorem (Direct sum). *Let Y be any closed subspace of a Hilbert space H. Then*

$$H = Y \oplus Z \qquad\qquad Z = Y^\perp. \tag{6}$$

Proof. Since H is complete and Y is closed, Y is complete by Theorem 1.4-7. Since Y is convex, Theorem 3.3-1 and Lemma 3.3-2 imply that for every $x \in H$ there is a $y \in Y$ such that

$$x = y + z \qquad z \in Z = Y^{\perp}. \tag{7}$$

To prove uniqueness, we assume that

$$x = y + z = y_1 + z_1$$

where $y, y_1 \in Y$ and $z, z_1 \in Z$. Then $y - y_1 = z_1 - z$. Since $y - y_1 \in Y$ whereas $z_1 - z \in Z = Y^{\perp}$, we see that $y - y_1 \in Y \cap Y^{\perp} = \{0\}$. This implies $y = y_1$. Hence also $z = z_1$. ∎

y in (7) is called the **orthogonal projection** of x on Y (or, briefly, the *projection* of x on Y). This term is motivated by elementary geometry. [For instance, we can take $H = \mathbf{R}^2$ and project any point $x = (\xi_1, \xi_2)$ on the ξ_1-axis, which then plays the role of Y; the projection is $y = (\xi_1, 0)$.]

Equation (7) defines a mapping

$$P: H \longrightarrow Y$$

$$x \longmapsto y = Px.$$

P is called the (orthogonal) **projection** (or *projection operator*) of H onto Y. See Fig. 29. Obviously, P is a bounded linear operator. P

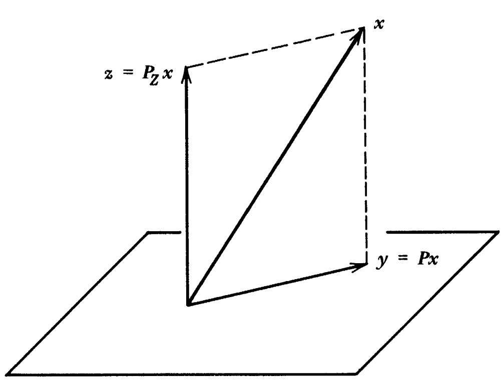

Fig. 29. Notation in connection with Theorem 3.3-4 and formula (9)

maps

$$\begin{aligned} &H \text{ onto } Y, \\ &Y \text{ onto itself}, \\ &Z = Y^{\perp} \text{ onto } \{0\}, \end{aligned}$$

and is **idempotent,** that is,

$$P^2 = P;$$

thus, for every $x \in H$,

$$P^2 x = P(Px) = Px.$$

Hence $P|_Y$ is the identity operator on Y. And for $Z = Y^{\perp}$ our discussion yields

3.3-5 Lemma (Null space). *The orthogonal complement $Y^{\perp}$ of a closed subspace Y of a Hilbert space H is the null space $\mathcal{N}(P)$ of the orthogonal projection P of H onto Y.*

An orthogonal complement is a special annihilator, where, by definition, the *annihilator* $M^{\perp}$ of a set $M \neq \varnothing$ in an inner product space X is the set[3]

$$M^{\perp} = \{x \in X \mid x \perp M\}.$$

Thus, $x \in M^{\perp}$ if and only if $\langle x, v \rangle = 0$ for all $v \in M$. This explains the name.

Note that $M^{\perp}$ is a vector space since $x, y \in M^{\perp}$ implies for all $v \in M$ and all scalars α, β

$$\langle \alpha x + \beta y, v \rangle = \alpha \langle x, v \rangle + \beta \langle y, v \rangle = 0,$$

hence $\alpha x + \beta y \in M^{\perp}$.

$M^{\perp}$ is closed, as the reader may prove (Prob. 8).

$(M^{\perp})^{\perp}$ is written $M^{\perp\perp}$, etc. In general we have

$$M \subset M^{\perp\perp} \tag{8*}$$

[3] This causes no conflict with Prob. 13, Sec. 2.10, as we shall see later (in Sec. 3.8).

because

$$x \in M \implies x \perp M^\perp \implies x \in (M^\perp)^\perp.$$

But for closed subspaces we even have

3.3-6 Lemma (Closed subspace). *If Y is a closed subspace of a Hilbert space H, then*

(8) $$Y = Y^{\perp\perp}.$$

Proof. $Y \subset Y^{\perp\perp}$ by (8*). We show $Y \supset Y^{\perp\perp}$. Let $x \in Y^{\perp\perp}$. Then $x = y + z$ by 3.3-4, where $y \in Y \subset Y^{\perp\perp}$ by (8*). Since $Y^{\perp\perp}$ is a vector space and $x \in Y^{\perp\perp}$ by assumption, we also have $z = x - y \in Y^{\perp\perp}$, hence $z \perp Y^\perp$. But $z \in Y^\perp$ by 3.3-4. Together $z \perp z$, hence $z = 0$, so that $x = y$, that is, $x \in Y$. Since $x \in Y^{\perp\perp}$ was arbitrary, this proves $Y \supset Y^{\perp\perp}$. ∎

(8) is the main reason for the use of *closed* subspaces in the present context. Since $Z^\perp = Y^{\perp\perp} = Y$, formula (6) can also be written

$$H = Z \oplus Z^\perp.$$

It follows that $x \longmapsto z$ defines a projection (Fig. 29)

(9) $$P_Z \colon H \longrightarrow Z$$

of H onto Z, whose properties are quite similar to those of the projection P considered before.

Theorem 3.3-4 readily implies a characterization of sets in Hilbert spaces whose span is dense, as follows.

3.3-7 Lemma (Dense set). *For any subset $M \neq \varnothing$ of a Hilbert space H, the span of M is dense in H if and only if $M^\perp = \{0\}$.*

Proof. (*a*) Let $x \in M^\perp$ and assume $V = \operatorname{span} M$ to be dense in H. Then $x \in \bar{V} = H$. By Theorem 1.4-6(*a*) there is a sequence (x_n) in V such that $x_n \longrightarrow x$. Since $x \in M^\perp$ and $M^\perp \perp V$, we have $\langle x_n, x\rangle = 0$. The continuity of the inner product (cf. Lemma 3.2-2) implies that $\langle x_n, x\rangle \longrightarrow \langle x, x\rangle$. Together, $\langle x, x\rangle = \|x\|^2 = 0$, so that $x = 0$. Since $x \in M^\perp$ was arbitrary, this shows that $M^\perp = \{0\}$.

(*b*) Conversely, suppose that $M^\perp = \{0\}$. If $x \perp V$, then $x \perp M$, so that $x \in M^\perp$ and $x = 0$. Hence $V^\perp = \{0\}$. Noting that V is a subspace of H, we thus obtain $\bar{V} = H$ from 3.3-4 with $Y = \bar{V}$. ∎

Problems

1. Let H be a Hilbert space, $M \subset H$ a convex subset, and (x_n) a sequence in M such that $\|x_n\| \longrightarrow d$, where $d = \inf_{x \in M} \|x\|$. Show that (x_n) converges in H. Give an illustrative example in $\mathbf{R}^2$ or $\mathbf{R}^3$.

2. Show that the subset $M = \{y = (\eta_j) \mid \sum \eta_j = 1\}$ of complex space $\mathbf{C}^n$ (cf. 3.1-4) is complete and convex. Find the vector of minimum norm in M.

3. (*a*) Show that the vector space X of all real-valued continuous functions on $[-1, 1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$. (*b*) Give examples of representations of $\mathbf{R}^3$ as a direct sum (*i*) of a subspace and its orthogonal complement, (*ii*) of any complementary pair of subspaces.

4. (*a*) Show that the conclusion of Theorem 3.3-1 also holds if X is a Hilbert space and $M \subset X$ is a closed subspace. (*b*) How could we use Appolonius' identity (Sec. 3.1, Prob. 5) in the proof of Theorem 3.3-1?

5. Let $X = \mathbf{R}^2$. Find $M^\perp$ if M is (*a*) $\{x\}$, where $x = (\xi_1, \xi_2) \neq 0$, (*b*) a linearly independent set $\{x_1, x_2\} \subset X$.

6. Show that $Y = \{x \mid x = (\xi_j) \in l^2, \ \xi_{2n} = 0, \ n \in \mathbf{N}\}$ is a closed subspace of l^2 and find $Y^\perp$. What is $Y^\perp$ if $Y = \operatorname{span}\{e_1, \cdots, e_n\} \subset l^2$, where $e_j = (\delta_{jk})$?

7. Let A and $B \supset A$ be nonempty subsets of an inner product space X. Show that

(*a*) $A \subset A^{\perp\perp}$, (*b*) $B^\perp \subset A^\perp$, (*c*) $A^{\perp\perp\perp} = A^\perp$.

8. Show that the annihilator $M^\perp$ of a set $M \neq \varnothing$ in an inner product space X is a closed subspace of X.

9. Show that a subspace Y of a Hilbert space H is closed in H if and only if $Y = Y^{\perp\perp}$.

10. If $M \neq \varnothing$ is any subset of a Hilbert space H, show that $M^{\perp\perp}$ is the smallest closed subspace of H which contains M, that is, $M^{\perp\perp}$ is contained in any closed subspace $Y \subset H$ such that $Y \supset M$.

3.4 Orthonormal Sets and Sequences

Orthogonality of elements as defined in Sec. 3.1 plays a basic role in inner product and Hilbert spaces. A first impression of this fact was given in the preceding section. Of particular interest are sets whose elements are orthogonal in pairs. To understand this, let us remember a familiar situation in Euclidean space $\mathbf{R}^3$. In the space $\mathbf{R}^3$, a set of that kind is the set of the three unit vectors in the positive directions of the axes of a rectangular coordinate system; call these vectors e_1, e_2, e_3. These vectors form a basis for $\mathbf{R}^3$, so that every $x \in \mathbf{R}^3$ has a unique representation (Fig. 30)

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3.$$

Now we see a great advantage of the orthogonality. Given x, we can readily determine the unknown coefficients α_1, α_2, α_3 by taking inner products (dot products). In fact, to obtain α_1, we must multiply that representation of x by e_1, that is,

$$\langle x, e_1 \rangle = \alpha_1 \langle e_1, e_1 \rangle + \alpha_2 \langle e_2, e_1 \rangle + \alpha_3 \langle e_3, e_1 \rangle = \alpha_1,$$

and so on. In more general inner product spaces there are similar and other possibilities for the use of orthogonal and orthonormal sets and

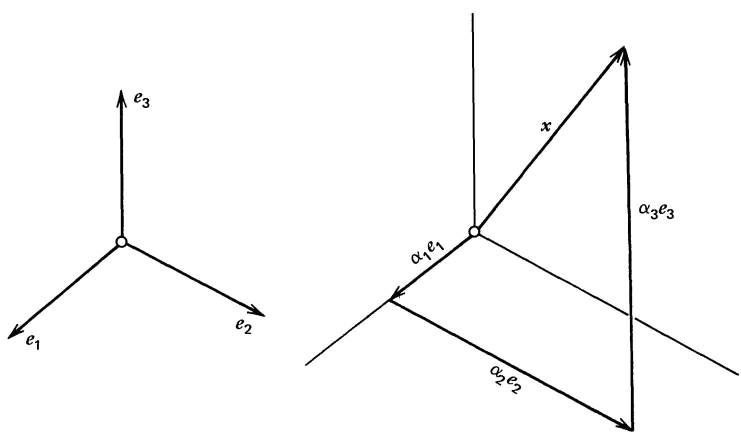

Fig. 30. Orthonormal set $\{e_1, e_2, e_3\}$ in $\mathbf{R}^3$ and representation $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

sequences, as we shall explain. In fact, the application of such sets and sequences makes up quite a substantial part of the whole theory of inner product and Hilbert spaces. Let us begin our study of this situation by introducing the necessary concepts.

3.4-1 Definition (Orthonormal sets and sequences). *An orthogonal set* M in an inner product space X is a subset $M \subset X$ whose elements are pairwise orthogonal. An *orthonormal set* $M \subset X$ is an orthogonal set in X whose elements have norm 1, that is, for all $x, y \in M$,

$$\text{(1)} \qquad \langle x, y\rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

If an orthogonal or orthonormal set M is countable, we can arrange it in a sequence (x_n) and call it an *orthogonal* or *orthonormal sequence*, respectively.

More generally, an indexed set, or *family*, (x_α), $\alpha \in I$, is called *orthogonal* if $x_\alpha \perp x_\beta$ for all $\alpha, \beta \in I$, $\alpha \neq \beta$. The family is called *orthonormal* if it is orthogonal and all x_α have norm 1, so that for all $\alpha, \beta \in I$ we have

$$\text{(2)} \qquad \langle x_\alpha, x_\beta\rangle = \delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta, as in Sec. 2.9. ∎

If the reader needs help with families and related concepts, he should look up A1.3 in Appendix 1. He will note that the concepts in our present definition are closely related. The reason is that to any subset M of X we can always find a family of elements of X such that the set of the elements of the family is M. In particular, we may take the family defined by the *natural injection* of M into X, that is, the restriction to M of the identity mapping $x \longmapsto x$ on X.

We shall now consider some simple properties and examples of orthogonal and orthonormal sets.

For orthogonal elements x, y we have $\langle x, y\rangle = 0$, so that we readily obtain the **Pythagorean relation**

$$\text{(3)} \qquad \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

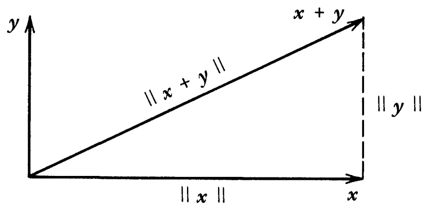

Fig. 31. Pythagorean relation (3) in $\mathbf{R}^2$

Figure 31 shows a familiar example.—More generally, if $\{x_1, \cdots, x_n\}$ is an orthogonal set, then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2. \tag{4}$$

In fact, $\langle x_j, x_k\rangle = 0$ if $j \neq k$; consequently,

$$\Big\|\sum_j x_j\Big\|^2 = \Big\langle \sum_j x_j, \sum_k x_k\Big\rangle = \sum_j \sum_k \langle x_j, x_k\rangle = \sum_j \langle x_j, x_j\rangle = \sum_j \|x_j\|^2$$

(summations from 1 to n). We also note

3.4-2 Lemma (Linear independence). *An orthonormal set is linearly independent.*

Proof. Let $\{e_1, \cdots, e_n\}$ be orthonormal and consider the equation

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0.$$

Multiplication by a fixed e_j gives

$$\Big\langle \sum_k \alpha_k e_k, e_j\Big\rangle = \sum_k \alpha_k \langle e_k, e_j\rangle = \alpha_j \langle e_j, e_j\rangle = \alpha_j = 0$$

and proves linear independence for any finite orthonormal set. This also implies linear independence if the given orthonormal set is infinite, by the definition of linear independence in Sec. 2.1. ∎

Examples

3.4-3 Euclidean space $\mathbf{R}^3$. In the space $\mathbf{R}^3$, the three unit vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ in the direction of the three axes of a rectangular coordinate system form an orthonormal set. See Fig. 30.

3.4-4 Space l^2. In the space l^2, an orthonormal sequence is (e_n), where $e_n = (\delta_{nj})$ has the nth element 1 and all others zero. (Cf. 3.1-6.)

3.4-5 Continuous functions. Let X be the inner product space of all real-valued continuous functions on $[0, 2\pi]$ with inner product defined by

$$\langle x, y \rangle = \int_0^{2\pi} x(t)y(t)\,dt$$

(cf. 3.1-5). An orthogonal sequence in X is (u_n), where

$$u_n(t) = \cos nt \qquad n = 0, 1, \cdots.$$

Another orthogonal sequence in X is (v_n), where

$$v_n(t) = \sin nt \qquad n = 1, 2, \cdots.$$

In fact, by integration we obtain

$$(5) \qquad \langle u_m, u_n \rangle = \int_0^{2\pi} \cos mt \cos nt\,dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n = 1, 2, \cdots \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

and similarly for (v_n). Hence an orthonormal sequence is (e_n), where

$$e_0(t) = \frac{1}{\sqrt{2\pi}}, \qquad e_n(t) = \frac{u_n(t)}{\|u_n\|} = \frac{\cos nt}{\sqrt{\pi}} \qquad (n = 1, 2, \cdots).$$

From (v_n) we obtain the orthonormal sequence $(\tilde{e}_n)$, where

$$\tilde{e}_n(t) = \frac{v_n(t)}{\|v_n\|} = \frac{\sin nt}{\sqrt{\pi}} \qquad (n = 1, 2, \cdots).$$

Note that we even have $u_m \perp v_n$ for all m and n. (Proof?) These sequences appear in *Fourier series*, as we shall discuss in the next section. Our examples are sufficient to give us a first impression of what is going on. Further orthonormal sequences of practical importance are included in a later section (Sec. 3.7). ∎

A great advantage of orthonormal sequences over arbitrary linearly independent sequences is the following. If we know that a given x can be represented as a linear combination of some elements of an orthonormal sequence, then the orthonormality makes the actual determination of the coefficients very easy. In fact, if $(e_1, e_2, \cdots)$ is an orthonormal sequence in an inner product space X and we have $x \in \operatorname{span}\{e_1, \cdots, e_n\}$, where n is fixed, then by the definition of the span (Sec. 2.1),

$$x = \sum_{k=1}^{n} \alpha_k e_k, \tag{6}$$

and if we take the inner product by a fixed e_j, we obtain

$$\langle x, e_j\rangle = \left\langle \sum \alpha_k e_k, e_j \right\rangle = \sum \alpha_k \langle e_k, e_j\rangle = \alpha_j.$$

With these coefficients, (6) becomes

$$x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k. \tag{7}$$

This shows that the determination of the unknown coefficients in (6) is simple. Another advantage of orthonormality becomes apparent if in (6) and (7) we want to add another term $\alpha_{n+1}e_{n+1}$, to take care of an

$$\tilde{x} = x + \alpha_{n+1}e_{n+1} \in \operatorname{span}\{e_1, \cdots, e_{n+1}\};$$

then we need to calculate only one more coefficient since the other coefficients remain unchanged.

More generally, if we consider any $x \in X$, not necessarily in $Y_n = \operatorname{span}\{e_1, \cdots, e_n\}$, we can define $y \in Y_n$ by setting

$$y = \sum_{k=1}^{n} \langle x, e_k\rangle e_k, \tag{8a}$$

where n is fixed, as before, and then define z by setting

$$x = y + z, \tag{8b}$$

that is, $z = x - y$. We want to show that $z \perp y$. To really understand what is going on, note the following. *Every* $y \in Y_n$ is a linear combination

$$y = \sum_{k=1}^{n} \alpha_k e_k.$$

Here $\alpha_k = \langle y, e_k \rangle$, as follows from what we discussed right before. Our claim is that for the particular choice $\alpha_k = \langle x, e_k \rangle$, $k = 1, \cdots, n$, we shall obtain a y such that $z = x - y \perp y$.

To prove this, we first note that, by the orthonormality,

$$\|y\|^2 = \left\langle \sum \langle x, e_k \rangle e_k, \sum \langle x, e_m \rangle e_m \right\rangle = \sum |\langle x, e_k \rangle|^2. \tag{9}$$

Using this, we can now show that $z \perp y$:

$$\begin{aligned} \langle z, y \rangle = \langle x - y, y \rangle &= \langle x, y \rangle - \langle y, y \rangle \\ &= \left\langle x, \sum \langle x, e_k \rangle e_k \right\rangle - \|y\|^2 \\ &= \sum \langle x, e_k \rangle \overline{\langle x, e_k \rangle} - \sum |\langle x, e_k \rangle|^2 \\ &= 0. \end{aligned}$$

Hence the Pythagorean relation (3) gives

$$\|x\|^2 = \|y\|^2 + \|z\|^2. \tag{10}$$

By (9) it follows that

$$\|z\|^2 = \|x\|^2 - \|y\|^2 = \|x\|^2 - \sum |\langle x, e_k \rangle|^2. \tag{11}$$

Since $\|z\| \geqq 0$, we have for every $n = 1, 2, \cdots$

$$\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \leqq \|x\|^2. \tag{12*}$$

These sums have nonnegative terms, so that they form a monotone increasing sequence. This sequence converges because it is bounded by $\|x\|^2$. This is the sequence of the partial sums of an infinite series, which thus converges. Hence (12*) implies

3.4-6 Theorem (Bessel inequality). *Let (e_k) be an orthonormal sequence in an inner product space X. Then for every $x \in X$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leqq \|x\|^2 \qquad \text{(12)} \quad \textbf{(Bessel inequality).}$$

The inner products $\langle x, e_k\rangle$ in (12) are called the **Fourier coefficients** of x with respect to the orthonormal sequence (e_k).

Note that if X is finite dimensional, then every orthonormal set in X must be finite because it is linearly independent by 3.4-2. Hence in (12) we then have a finite sum.

We have seen that orthonormal sequences are very convenient to work with. The remaining practical problem is how to obtain an orthonormal sequence if an arbitrary linearly independent sequence is given. This is accomplished by a constructive procedure, the **Gram-Schmidt process** for orthonormalizing a linearly independent sequence (x_j) in an inner product space. The resulting orthonormal sequence (e_j) has the property that for every n,

$$\operatorname{span}\{e_1, \cdots, e_n\} = \operatorname{span}\{x_1, \cdots, x_n\}.$$

The process is as follows.

1st step. The first element of (e_k) is

$$e_1 = \frac{1}{\|x_1\|} x_1.$$

2nd step. x_2 can be written

$$x_2 = \langle x_2, e_1\rangle e_1 + v_2.$$

Then (Fig. 32)

$$v_2 = x_2 - \langle x_2, e_1\rangle e_1$$

is not the zero vector since (x_j) is linearly independent; also $v_2 \perp e_1$ since $\langle v_2, e_1\rangle = 0$, so that we can take

$$e_2 = \frac{1}{\|v_2\|} v_2.$$

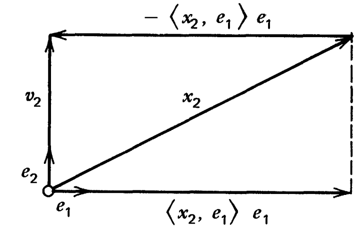

Fig. 32. Gram-Schmidt process, 2nd step

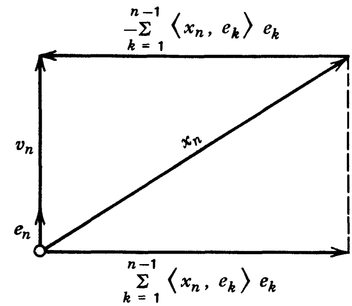

Fig. 33. Gram-Schmidt process, nth step

3rd step. The vector

$$v_3 = x_3 - \langle x_3, e_1\rangle e_1 - \langle x_3, e_2\rangle e_2$$

is not the zero vector, and $v_3 \perp e_1$ as well as $v_3 \perp e_2$. We take

$$e_3 = \frac{1}{\|v_3\|} v_3.$$

***n*th step.** The vector (see Fig. 33)

$$v_n = x_n - \sum_{k=1}^{n-1} \langle x_n, e_k\rangle e_k \tag{13}$$

is not the zero vector and is orthogonal to $e_1, \cdots, e_{n-1}$. From it we obtain

$$e_n = \frac{1}{\|v_n\|} v_n. \tag{14}$$

These are the general formulas for the Gram-Schmidt process, which was designed by E. Schmidt (1907). Cf. also J. P. Gram (1883). Note that the sum which is subtracted on the right-hand side of (13) is the projection of x_n on span $\{e_1, \cdots, e_{n-1}\}$. In other words, in each step we subtract from x_n its "components" in the directions of the previously orthogonalized vectors. This gives v_n, which is then multiplied by $1/\|v_n\|$, so that we get a vector of norm one. v_n cannot be the zero vector for any n. In fact, if n were the smallest subscript for which $v_n = 0$, then (13) shows that x_n would be a linear combination of $e_1, \cdots, e_{n-1}$, hence a linear combination of $x_1, \cdots, x_{n-1}$, contradicting the assumption that $\{x_1, \cdots, x_n\}$ is linearly independent.

Problems

1. Show that an inner product space of finite dimension n has a basis $\{b_1, \cdots, b_n\}$ of orthonormal vectors. (The infinite dimensional case will be considered in Sec. 3.6.)

2. How can we interpret (12*) geometrically in $\mathbf{R}^r$, where $r \geqq n$?

3. Obtain the Schwarz inequality (Sec. 3.2) from (12*).

4. Give an example of an $x \in l^2$ such that we have strict inequality in (12).

5. If (e_k) is an orthonormal sequence in an inner product space X, and $x \in X$, show that $x - y$ with y given by

$$y = \sum_{k=1}^{n} \alpha_k e_k \qquad \alpha_k = \langle x, e_k \rangle$$

is orthogonal to the subspace $Y_n = \operatorname{span}\{e_1, \cdots e_n\}$.

6. **(Minimum property of Fourier coefficients)** Let $\{e_1, \cdots, e_n\}$ be an orthonormal set in an inner product space X, where n is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \cdots + \beta_n e_n$. Then $\|x - y\|$ depends on $\beta_1, \cdots, \beta_n$. Show by direct calculation that $\|x - y\|$ is minimum if and only if $\beta_j = \langle x, e_j \rangle$, where $j = 1, \cdots, n$.

7. Let (e_k) be any orthonormal sequence in an inner product space X. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leqq \|x\| \, \|y\|.$$

8. Show that an element x of an inner product space X cannot have "too many" Fourier coefficients $\langle x, e_k \rangle$ which are "big"; here, (e_k) is a given orthonormal sequence; more precisely, show that the number n_m of $\langle x, e_k \rangle$ such that $|\langle x, e_k \rangle| > 1/m$ must satisfy $n_m < m^2 \|x\|^2$.

9. Orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, \cdots)$, where $x_j(t) = t^j$, on the interval $[-1, 1]$, where

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)\, dt.$$

10. Let $x_1(t) = t^2$, $x_2(t) = t$ and $x_3(t) = 1$. Orthonormalize x_1, x_2, x_3, in this order, on the interval $[-1, 1]$ with respect to the inner product given in Prob. 9. Compare with Prob. 9 and comment.

3.5 Series Related to Orthonormal Sequences and Sets

There are some facts and questions that arise in connection with the Bessel inequality. In this section we first motivate the term "Fourier coefficients," then consider infinite series related to orthonormal sequences, and finally take a first look at orthonormal sets which are uncountable.

3.5-1 Example (Fourier series). A *trigonometric series* is a series of the form

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt). \tag{1*}$$

A real-valued function x on $\mathbf{R}$ is said to be *periodic* if there is a positive number p (called a *period* of x) such that $x(t+p) = x(t)$ for all $t \in \mathbf{R}$.

Let x be of period 2π and continuous. By definition, the *Fourier series* of x is the trigonometric series (1*) with coefficients a_k and b_k given by the *Euler formulas*

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt \\ a_k &= \frac{1}{\pi} \int_0^{2\pi} x(t) \cos kt\, dt \qquad k = 1, 2, \cdots, \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} x(t) \sin kt\, dt \qquad k = 1, 2, \cdots. \end{aligned} \tag{2}$$

These coefficients are called the *Fourier coefficients of x*.

If the Fourier series of x converges for each t and has the sum $x(t)$, then we write

(1) $$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt).$$

Since x is periodic of period 2π, in (2) we may replace the interval of integration $[0, 2\pi]$ by any other interval of length 2π, for instance $[-\pi, \pi]$.

Fourier series first arose in connection with physical problems considered by D. Bernoulli (vibrating string, 1753) and J. Fourier (heat conduction, 1822). These series help to represent complicated periodic phenomena in terms of simple periodic functions (cosine and sine). They have various physical applications in connection with differential equations (vibrations, heat conduction, potential problems, etc.).

From (2) we see that the determination of Fourier coefficients requires integration. To help those readers who have not seen Fourier series before, we consider as an illustration (see Fig. 34)

$$x(t) = \begin{cases} t & \text{if } -\pi/2 \leqq t < \pi/2 \\ \pi - t & \text{if } \pi/2 \leqq t < 3\pi/2 \end{cases}$$

and $x(t+2\pi) = x(t)$. From (2) we obtain $a_k = 0$ for $k = 0, 1, \cdots$ and, choosing $[-\pi/2, 3\pi/2]$ as a convenient interval of integration and integrating by parts,

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} t \sin kt\, dt + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} (\pi - t) \sin kt\, dt \\ &= -\frac{1}{\pi k} [t \cos kt] \Big|_{-\pi/2}^{\pi/2} + \frac{1}{\pi k} \int_{-\pi/2}^{\pi/2} \cos kt\, dt \\ &\quad - \frac{1}{\pi k} [(\pi - t) \cos kt] \Big|_{\pi/2}^{3\pi/2} - \frac{1}{\pi k} \int_{\pi/2}^{3\pi/2} \cos kt\, dt \\ &= \frac{4}{\pi k^2} \sin \frac{k\pi}{2}, \qquad k = 1, 2, \cdots. \end{aligned}$$

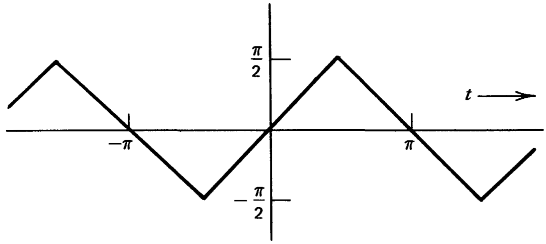

Fig. 34. Graph of the periodic function x, of period 2π, given by $x(t)=t$ if $t\in[-\pi/2, \pi/2)$ and $x(t)=\pi-t$ if $t\in[\pi/2, 3\pi/2)$

Hence (1) takes the form

$$x(t)=\frac{4}{\pi}\left(\sin t-\frac{1}{3^2}\sin 3t+\frac{1}{5^2}\sin 5t-+\cdots\right).$$

The reader may graph the first three partial sums and compare them with the graph of x in Fig. 34.

Returning to general Fourier series, we may ask how these series fit into our terminology and formalism introduced in the preceding section. Obviously, the cosine and sine functions in (1) are those of the sequences (u_k) and (v_k) in 3.4-5, that is

$$u_k(t)=\cos kt, \qquad v_k(t)=\sin kt.$$

Hence we may write (1) in the form

$$x(t)=a_0u_0(t)+\sum_{k=1}^{\infty}[a_ku_k(t)+b_kv_k(t)]. \tag{3}$$

We multiply (3) by a fixed u_j and integrate over t from 0 to 2π. This means that we take the inner product by u_j as defined in 3.4-5. We assume that termwise integration is permissible (uniform convergence would suffice) and use the orthogonality of (u_k) and (v_k) as well as the fact that $u_j \perp v_k$ for all j, k. Then we obtain

$$\begin{aligned}\langle x, u_j\rangle &= a_0\langle u_0, u_j\rangle+\sum[a_k\langle u_k, u_j\rangle+b_k\langle v_k, u_j\rangle]\\ &= a_j\langle u_j, u_j\rangle\\ &= a_j\,\|u_j\|^2=\begin{cases}2\pi a_0 & \text{if } j=0\\ \pi a_j & \text{if } j=1, 2, \cdots,\end{cases}\end{aligned}$$

cf. (5), Sec. 3.4. Similarly, if we multiply (3) by v_j and proceed as before, we arrive at

$$\langle x, v_j \rangle = b_j \|v_j\|^2 = \pi b_j$$

where $j = 1, 2, \cdots$. Solving for a_j and b_j and using the orthonormal sequences (e_j) and $(\tilde{e}_j)$, where $e_j = \|u_j\|^{-1} u_j$ and $\tilde{e}_j = \|v_j\|^{-1} v_j$, we obtain

$$\begin{aligned} a_j &= \frac{1}{\|u_j\|^2} \langle x, u_j \rangle = \frac{1}{\|u_j\|} \langle x, e_j \rangle, \\ b_j &= \frac{1}{\|v_j\|^2} \langle x, v_j \rangle = \frac{1}{\|v_j\|} \langle x, \tilde{e}_j \rangle. \end{aligned} \tag{4}$$

This is identical with (2). It shows that in (3),

$$a_k u_k(t) = \frac{1}{\|u_k\|} \langle x, e_k \rangle u_k(t) = \langle x, e_k \rangle e_k(t)$$

and similarly for $b_k v_k(t)$. Hence we may write the Fourier series (1) in the form

$$x = \langle x, e_0 \rangle e_0 + \sum_{k=1}^{\infty} [\langle x, e_k \rangle e_k + \langle x, \tilde{e}_k \rangle \tilde{e}_k]. \tag{5}$$

This justifies the term "Fourier coefficients" in the preceding section.

Concluding this example, we mention that the reader can find an introduction to Fourier series in W. Rogosinski (1959); cf. also R. V. Churchill (1963), pp. 77–112 and E. Kreyszig (1972), pp. 377–407. ∎

Our example concerns infinite series and raises the question how we can extend the consideration to other orthonormal sequences and what we can say about the convergence of corresponding series.

Given any orthonormal sequence (e_k) in a Hilbert space H, we may consider series of the form

$$\sum_{k=1}^{\infty} \alpha_k e_k \tag{6}$$

where $\alpha_1, \alpha_2, \cdots$ are any scalars. As defined in Sec. 2.3, such a series *converges* and has the *sum* s if there exists an $s \in H$ such that the sequence (s_n) of the partial sums

$$s_n = \alpha_1 e_1 + \cdots + \alpha_n e_n$$

converges to s, that is, $\|s_n - s\| \longrightarrow 0$ as $n \longrightarrow \infty$.

3.5-2 Theorem (Convergence). *Let (e_k) be an orthonormal sequence in a Hilbert space H. Then:*

(a) *The series (6) converges (in the norm on H) if and only if the following series converges:*

(7)
$$\sum_{k=1}^{\infty} |\alpha_k|^2.$$

(b) *If (6) converges, then the coefficients α_k are the Fourier coefficients $\langle x, e_k \rangle$, where x denotes the sum of (6); hence in this case, (6) can be written*

(8)
$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k.$$

(c) *For any $x \in H$, the series (6) with $\alpha_k = \langle x, e_k \rangle$ converges (in the norm of H).*

Proof. **(a)** Let

$$s_n = \alpha_1 e_1 + \cdots + \alpha_n e_n \qquad \text{and} \qquad \sigma_n = |\alpha_1|^2 + \cdots + |\alpha_n|^2.$$

Then, because of the orthonormality, for any m and $n > m$,

$$\begin{aligned} \|s_n - s_m\|^2 &= \|\alpha_{m+1} e_{m+1} + \cdots + \alpha_n e_n\|^2 \\ &= |\alpha_{m+1}|^2 + \cdots + |\alpha_n|^2 = \sigma_n - \sigma_m. \end{aligned}$$

Hence (s_n) is Cauchy in H if and only if (σ_n) is Cauchy in **R**. Since H and **R** are complete, the first statement of the theorem follows.

(b) Taking the inner product of s_n and e_j and using the orthonormality, we have

$$\langle s_n, e_j \rangle = \alpha_j \quad \text{for } j = 1, \cdots, k \qquad (k \leqq n \text{ and fixed}).$$

By assumption, $s_n \longrightarrow x$. Since the inner product is continuous (cf. Lemma 3.2-2),

$$\alpha_j = \langle s_n, e_j \rangle \longrightarrow \langle x, e_j \rangle \qquad (j \leqq k).$$

Here we can take k ($\leqq n$) as large as we please because $n \longrightarrow \infty$, so that we have $\alpha_j = \langle x, e_j \rangle$ for every $j = 1, 2, \cdots$.

(c) From the Bessel inequality in Theorem 3.4-6 we see that the series

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

converges. From this and (*a*) we conclude that (*c*) must hold. ∎

If an orthonormal family (e_κ), $\kappa \in I$, in an inner product space X is uncountable (since the index set I is uncountable), we can still form the Fourier coefficients $\langle x, e_\kappa \rangle$ of an $x \in X$, where $\kappa \in I$. Now we use (12*), Sec. 3.4, to conclude that for each fixed $m = 1, 2, \cdots$ the number of Fourier coefficients such that $|\langle x, e_\kappa \rangle| > 1/m$ must be finite. This proves the remarkable

3.5-3 Lemma (Fourier coefficients). *Any x in an inner product space X can have at most countably many nonzero Fourier coefficients $\langle x, e_\kappa \rangle$ with respect to an orthonormal family (e_κ), $\kappa \in I$, in X.*

Hence with any fixed $x \in H$ we can associate a series similar to (8),

$$\sum_{\kappa \in I} \langle x, e_\kappa \rangle e_\kappa \tag{9}$$

and we can arrange the e_κ with $\langle x, e_\kappa \rangle \neq 0$ in a sequence $(e_1, e_2, \cdots)$, so that (9) takes the form (8). Convergence follows from Theorem 3.5-2. We show that the sum does not depend on the order in which those e_κ are arranged in a sequence.

Proof. Let (w_m) be a rearrangement of (e_n). By definition this means that there is a bijective mapping $n \longmapsto m(n)$ of $\mathbf{N}$ onto itself such that corresponding terms of the two sequences are equal, that is,

$w_{m(n)} = e_n$. We set

$$\alpha_n = \langle x, e_n \rangle, \qquad \beta_m = \langle x, w_m \rangle$$

and

$$x_1 = \sum_{n=1}^{\infty} \alpha_n e_n, \qquad x_2 = \sum_{m=1}^{\infty} \beta_m w_m.$$

Then by Theorem 3.5-2(*b*),

$$\alpha_n = \langle x, e_n \rangle = \langle x_1, e_n \rangle, \qquad \beta_m = \langle x, w_m \rangle = \langle x_2, w_m \rangle.$$

Since $e_n = w_{m(n)}$, we thus obtain

$$\begin{aligned} \langle x_1 - x_2, e_n \rangle &= \langle x_1, e_n \rangle - \langle x_2, w_{m(n)} \rangle \\ &= \langle x, e_n \rangle - \langle x, w_{m(n)} \rangle = 0 \end{aligned}$$

and similarly $\langle x_1 - x_2, w_m \rangle = 0$. This implies

$$\begin{aligned} \|x_1 - x_2\|^2 &= \langle x_1 - x_2, \sum \alpha_n e_n - \sum \beta_m w_m \rangle \\ &= \sum \bar{\alpha}_n \langle x_1 - x_2, e_n \rangle - \sum \bar{\beta}_m \langle x_1 - x_2, w_m \rangle = 0. \end{aligned}$$

Consequently, $x_1 - x_2 = 0$ and $x_1 = x_2$. Since the rearrangement (w_m) of (e_n) was arbitrary, this completes the proof. ∎

Problems

1. If (6) converges with sum x, show that (7) has the sum $\|x\|^2$.

2. Derive from (1) and (2) a Fourier series representation of a function $\tilde{x}$ (function of τ) of arbitrary period p.

3. Illustrate with an example that a convergent series $\sum \langle x, e_k \rangle e_k$ need not have the sum x.

4. If (x_j) is a sequence in an inner product space X such that the series $\|x_1\| + \|x_2\| + \cdots$ converges, show that (s_n) is a Cauchy sequence, where $s_n = x_1 + \cdots + x_n$.

5. Show that in a Hilbert space H, convergence of $\sum \|x_j\|$ implies convergence of $\sum x_j$.

6. Let (e_j) be an orthonormal sequence in a Hilbert space H. Show that if

$$x=\sum_{j=1}^{\infty} \alpha_j e_j, \qquad y=\sum_{j=1}^{\infty} \beta_j e_j, \qquad \text{then} \qquad \langle x, y\rangle=\sum_{j=1}^{\infty} \alpha_j \bar{\beta}_j,$$

the series being absolutely convergent.

7. Let (e_k) be an orthonormal sequence in a Hilbert space H. Show that for every $x \in H$, the vector

$$y=\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$$

exists in H and $x-y$ is orthogonal to every e_k.

8. Let (e_k) be an orthonormal sequence in a Hilbert space H, and let $M=\operatorname{span}(e_k)$. Show that for any $x \in H$ we have $x \in \bar{M}$ if and only if x can be represented by (6) with coefficients $\alpha_k=\langle x, e_k\rangle$.

9. Let (e_n) and $(\tilde{e}_n)$ be orthonormal sequences in a Hilbert space H, and let $M_1=\operatorname{span}(e_n)$ and $M_2=\operatorname{span}(\tilde{e}_n)$. Using Prob. 8, show that $\bar{M}_1=\bar{M}_2$ if and only if

$$\text{(a)} \quad e_n=\sum_{m=1}^{\infty} \alpha_{nm} \tilde{e}_m, \qquad \text{(b)} \quad \tilde{e}_n=\sum_{m=1}^{\infty} \bar{\alpha}_{mn} e_m, \qquad \alpha_{nm}=\langle e_n, \tilde{e}_m\rangle.$$

10. Work out the details of the proof of Lemma 3.5-3.

3.6 Total Orthonormal Sets and Sequences

The truly interesting orthonormal sets in inner product spaces and Hilbert spaces are those which consist of "sufficiently many" elements so that every element in space can be represented or sufficiently accurately approximated by the use of those orthonormal sets. In finite dimensional (n-dimensional) spaces the situation is simple; all we need is an orthonormal set of n elements. The question is what can be done to take care of infinite dimensional spaces, too. Relevant concepts are as follows.

3.6-1 Definition (Total orthonormal set). A *total set* (or *fundamental set*) in a normed space X is a subset $M \subset X$ whose span is dense in X (cf. 1.3-5). Accordingly, an orthonormal set (or sequence or family) in an inner product space X which is total in X is called a *total orthonormal set*[4] (or sequence or family, respectively) in X. ▮

M is total in X if and only if

$$\overline{\operatorname{span} M} = X.$$

This is obvious from the definition.

A total orthonormal family in X is sometimes called an *orthonormal basis* for X. However, it is important to note that this is not a basis, in the sense of algebra, for X as a vector space, unless X is finite dimensional.

In every Hilbert space $H \neq \{0\}$ there exists a total orthonormal set.

For a finite dimensional H this is clear. For an infinite dimensional separable H (cf. 1.3-5) it follows from the Gram-Schmidt process by (ordinary) induction. For a nonseparable H a (nonconstructive) proof results from Zorn's lemma, as we shall see in Sec. 4.1 where we introduce and explain the lemma for another purpose.

All total orthonormal sets in a given Hilbert space $H \neq \{0\}$ have the same cardinality. The latter is called the *Hilbert dimension* or *orthogonal dimension* of H. (If $H = \{0\}$, this dimension is defined to be 0.)

For a finite dimensional H the statement is clear since then the Hilbert dimension is the dimension in the sense of algebra. For an infinite dimensional separable H the statement will readily follow from Theorem 3.6-4 (below) and for a general H the proof would require somewhat more advanced tools from set theory; cf. E. Hewitt and K. Stromberg (1969), p. 246.

[4] Sometimes a *complete* orthonormal set, but we use "complete" only in the sense of Def. 1.4-3; this is preferable since we then avoid the use of the same word in connection with two entirely different concepts. [Moreover, some authors mean by "completeness" of an orthonormal set M the property expressed by (1) in Theorem 3.6-2. We do not adopt this terminology either.]

The following theorem shows that a total orthonormal set cannot be augmented to a more extensive orthonormal set by the adjunction of new elements.

3.6-2 Theorem (Totality). *Let M be a subset of an inner product space X. Then:*

(a) *If M is total in X, then there does not exist a nonzero $x \in X$ which is orthogonal to every element of M; briefly,*

$$x \perp M \quad \Longrightarrow \quad x = 0. \tag{1}$$

(b) *If X is complete, that condition is also sufficient for the totality of M in X.*

Proof. **(a)** Let H be the completion of X; cf. 3.2-3. Then X, regarded as a subspace of H, is dense in H. By assumption, M is total in X, so that span M is dense in X, hence dense in H. Lemma 3.3-7 now implies that the orthogonal complement of M in H is $\{0\}$. A fortiori, if $x \in X$ and $x \perp M$, then $x = 0$.

(b) If X is a Hilbert space and M satisfies that condition, so that $M^{\perp} = \{0\}$, then Lemma 3.3-7 implies that M is total in X. ∎

The completeness of X in (b) is essential. If X is not complete, there may not exist an orthonormal set $M \subset X$ such that M is total in X. An example was given by J. Dixmier (1953). Cf. also N. Bourbaki (1955), p. 155.

Another important criterion for totality can be obtained from the Bessel inequality (cf. 3.4-6). For this purpose we consider any given orthonormal set M in a Hilbert space H. From Lemma 3.5-3 we know that each fixed $x \in H$ has at most countably many nonzero Fourier coefficients, so that we can arrange these coefficients in a sequence, say, $\langle x, e_1 \rangle$, $\langle x, e_2 \rangle$, $\cdots$. The Bessel inequality is (cf. 3.4-6)

$$\sum_k |\langle x, e_k \rangle|^2 \leqq \|x\|^2 \tag{2}$$ **(Bessel inequality)**

where the left-hand side is an infinite series or a finite sum. With the equality sign this becomes

$$\sum_k |\langle x, e_k\rangle|^2 = \|x\|^2 \qquad \textbf{(Parseval relation)} \tag{3}$$

and yields another criterion for totality:

3.6-3 Theorem (Totality). *An orthonormal set M in a Hilbert space H is total in H if and only if for all $x \in H$ the Parseval relation* (3) *holds* (*summation over all nonzero Fourier coefficients of x with respect to M*).

Proof. (*a*) If M is not total, by Theorem 3.6-2 there is a nonzero $x \perp M$ in H. Since $x \perp M$, in (3) we have $\langle x, e_k\rangle = 0$ for all k, so that the left-hand side in (3) is zero, whereas $\|x\|^2 \neq 0$. This shows that (3) does not hold. Hence if (3) holds for all $x \in H$, then M must be total in H.

(*b*) Conversely, assume M to be total in H. Consider any $x \in H$ and its nonzero Fourier coefficients (cf. 3.5-3) arranged in a sequence $\langle x, e_1\rangle$, $\langle x, e_2\rangle$, $\cdots$, or written in some definite order if there are only finitely many of them. We now define y by

$$y = \sum_k \langle x, e_k\rangle e_k, \tag{4}$$

noting that in the case of an infinite series, convergence follows from Theorem 3.5-2. Let us show that $x - y \perp M$. For every e_j occurring in (4) we have, using the orthonormality,

$$\langle x - y, e_j\rangle = \langle x, e_j\rangle - \sum_k \langle x, e_k\rangle\langle e_k, e_j\rangle = \langle x, e_j\rangle - \langle x, e_j\rangle = 0.$$

And for every $v \in M$ not contained in (4) we have $\langle x, v\rangle = 0$, so that

$$\langle x - y, v\rangle = \langle x, v\rangle - \sum_k \langle x, e_k\rangle\langle e_k, v\rangle = 0 - 0 = 0.$$

Hence $x - y \perp M$, that is, $x - y \in M^{\perp}$. Since M is total in H, we have $M^{\perp} = \{0\}$ from 3.3-7. Together, $x - y = 0$, that is, $x = y$. Using (4) and again the orthonormality, we thus obtain (3) from

$$\|x\|^2 = \left\langle \sum_k \langle x, e_k \rangle e_k, \sum_m \langle x, e_m \rangle e_m \right\rangle = \sum_k \langle x, e_k \rangle \overline{\langle x, e_k \rangle}.$$

This completes the proof. ∎

Let us turn to Hilbert spaces which are separable. By Def. 1.3-5 such a space has a countable subset which is dense in the space. Separable Hilbert spaces are simpler than nonseparable ones since they cannot contain uncountable orthonormal sets:

3.6-4 Theorem (Separable Hilbert spaces). *Let H be a Hilbert space. Then:*

(a) *If H is separable, every orthonormal set in H is countable.*

(b) *If H contains an orthonormal sequence which is total in H, then H is separable.*

Proof. **(a)** Let H be separable, B any dense set in H and M any orthonormal set. Then any two distinct elements x and y of M have distance $\sqrt{2}$ since

$$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle + \langle y, y \rangle = 2.$$

Hence spherical neighborhoods N_x of x and N_y of y of radius $\sqrt{2}/3$ are disjoint. Since B is dense in H, there is a $b \in B$ in N_x and a $\tilde{b} \in B$ in N_y and $b \neq \tilde{b}$ since $N_x \cap N_y = \varnothing$. Hence if M were uncountable, we would have uncountably many such pairwise disjoint spherical neighborhoods (for each $x \in M$ one of them), so that B would be uncountable. Since B was any dense set, this means that H would not contain a dense set which is countable, contradicting separability. From this we conclude that M must be countable.

(b) Let (e_k) be a total orthonormal sequence in H and A the set of all linear combinations

$$\gamma_1^{(n)} e_1 + \cdots + \gamma_n^{(n)} e_n \qquad n = 1, 2, \cdots$$

where $\gamma_k^{(n)} = a_k^{(n)} + i b_k^{(n)}$ and $a_k^{(n)}$ and $b_k^{(n)}$ are rational (and $b_k^{(n)} = 0$ if H is real). Clearly, A is countable. We prove that A is dense in H by showing that for every $x \in H$ and $\varepsilon > 0$ there is a $v \in A$ such that $\|x - v\| < \varepsilon$.

Since the sequence (e_k) is total in H, there is an n such that $Y_n = \operatorname{span}\{e_1, \cdots, e_n\}$ contains a point whose distance from x is less than $\varepsilon/2$. In particular, $\|x - y\| < \varepsilon/2$ for the orthogonal projection y of x on Y_n, which is given by [cf. (8), Sec. 3.4]

$$y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k.$$

Hence we have

$$\left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\| < \frac{\varepsilon}{2}.$$

Since the rationals are dense on $\mathbf{R}$, for each $\langle x, e_k \rangle$ there is a $\gamma_k^{(n)}$ (with rational real and imaginary parts) such that

$$\left\| \sum_{k=1}^{n} [\langle x, e_k \rangle - \gamma_k^{(n)}] e_k \right\| < \frac{\varepsilon}{2}.$$

Hence $v \in A$ defined by

$$v = \sum_{k=1}^{n} \gamma_k^{(n)} e_k$$

satisfies

$$\begin{aligned} \|x - v\| &= \|x - \sum \gamma_k^{(n)} e_k\| \\ &\leqq \|x - \sum \langle x, e_k \rangle e_k\| + \|\sum \langle x, e_k \rangle e_k - \sum \gamma_k^{(n)} e_k\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that A is dense in H, and since A is countable, H is separable. ∎

For using Hilbert spaces in applications one must know what total orthonormal set or sets to choose in a specific situation and how to investigate properties of the elements of such sets. For certain function spaces this problem will be considered in the next section, which includes special functions of practical interest that arise in this context and have been investigated in very great detail. To conclude this section, let us point out that our present discussion has some further consequences which are of basic importance and can be formulated in terms of isomorphisms of Hilbert spaces. For this purpose we first remember from Sec. 3.2 the following.

An **isomorphism** of a Hilbert space H onto a Hilbert space $\tilde{H}$ over the same field is a bijective linear operator $T: H \longrightarrow \tilde{H}$ such that for all $x, y \in H$,

$$\langle Tx, Ty \rangle = \langle x, y \rangle. \tag{5}$$

H and $\tilde{H}$ are then called *isomorphic Hilbert spaces.* Since T is linear, it preserves the vector space structure, and (5) shows that T is isometric. From this and the bijectivity of T it follows that H and $\tilde{H}$ are algebraically as well as metrically indistinguishable; they are essentially the same, except for the nature of their elements, so that we may think of $\tilde{H}$ as being essentially H with a "tag" T attached to each vector x. Or we may regard H and $\tilde{H}$ as two copies (models) of the same abstract space, just as we often do in the case of n-dimensional Euclidean space.

Most exciting in this discussion is the fact that for each Hilbert dimension (cf. at the beginning of this section) there is just one abstract real Hilbert space and just one abstract complex Hilbert space. In other words, two abstract Hilbert spaces over the same field are distinguished only by their Hilbert dimension, a situation which generalizes that in the case of Euclidean spaces. This is the meaning of the following theorem.

3.6-5 Theorem (Isomorphism and Hilbert dimension). *Two Hilbert spaces H and $\tilde{H}$, both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.*

Proof. (*a*) If H is isomorphic with $\tilde{H}$ and $T: H \longrightarrow \tilde{H}$ is an isomorphism, then (5) shows that orthonormal elements in H have orthonormal images under T. Since T is bijective, we thus conclude that T maps every total orthonormal set in H onto a total orthonormal set in $\tilde{H}$. Hence H and $\tilde{H}$ have the same Hilbert dimension.

(*b*) Conversely, suppose that H and $\tilde{H}$ have the same Hilbert dimension. The case $H = \{0\}$ and $\tilde{H} = \{0\}$ is trivial. Let $H \neq \{0\}$. Then $\tilde{H} \neq \{0\}$, and any total orthonormal sets M in H and $\tilde{M}$ in $\tilde{H}$ have the same cardinality, so that we can index them by the same index set $\{k\}$ and write $M=(e_k)$ and $\tilde{M}=(\tilde{e}_k)$.

To show that H and $\tilde{H}$ are isomorphic, we construct an isomorphism of H onto $\tilde{H}$. For every $x \in H$ we have

$$x=\sum_k \langle x, e_k\rangle e_k \tag{6}$$

where the right-hand side is a finite sum or an infinite series (cf. 3.5-3), and $\sum_k |\langle x, e_k\rangle|^2<\infty$ by the Bessel inequality. Defining

$$\tilde{x}=Tx=\sum_k \langle x, e_k\rangle \tilde{e}_k \tag{7}$$

we thus have convergence by 3.5-2, so that $\tilde{x} \in \tilde{H}$. The operator T is linear since the inner product is linear with respect to the first factor. T is isometric, because by first using (7) and then (6) we obtain

$$\|\tilde{x}\|^2=\|Tx\|^2=\sum_k |\langle x, e_k\rangle|^2=\|x\|^2.$$

From this and (9), (10) in Sec. 3.1 we see that T preserves the inner product. Furthermore, isometry implies injectivity. In fact, if $Tx=Ty$, then

$$\|x-y\|=\|T(x-y)\|=\|Tx-Ty\|=0,$$

so that $x=y$ and T is injective by 2.6-10.

We finally show that T is surjective. Given any

$$\tilde{x}=\sum_k \alpha_k \tilde{e}_k$$

in $\tilde{H}$, we have $\sum |\alpha_k|^2<\infty$ by the Bessel inequality. Hence

$$\sum_k \alpha_k e_k$$

is a finite sum or a series which converges to an $x \in H$ by 3.5-2, and $\alpha_k=\langle x, e_k\rangle$ by the same theorem. We thus have $\tilde{x}=Tx$ by (7). Since $\tilde{x} \in \tilde{H}$ was arbitrary, this shows that T is surjective. ∎

Problems

1. If F is an orthonormal basis in an inner product space X, can we represent every $x \in X$ as a linear combination of elements of F? (By definition, a linear combination consists of finitely many terms.)

2. Show that if the orthogonal dimension of a Hilbert space H is finite, it equals the dimension of H regarded as a vector space; conversely, if the latter is finite, show that so is the former.

3. From what theorem of elementary geometry does (3) follow in the case of Euclidean n-space?

4. Derive from (3) the following formula (which is often called the *Parseval relation*).

$$\langle x, y\rangle = \sum_k \langle x, e_k\rangle \overline{\langle y, e_k\rangle}.$$

5. Show that an orthonormal family (e_κ), $\kappa \in I$, in a Hilbert space H is total if and only if the relation in Prob. 4 holds for every x and y in H.

6. Let H be a separable Hilbert space and M a countable dense subset of H. Show that H contains a total orthonormal sequence which can be obtained from M by the Gram-Schmidt process.

7. Show that if a Hilbert space H is separable, the existence of a total orthonormal set in H can be proved without the use of Zorn's lemma.

8. Show that for any orthonormal sequence F in a separable Hilbert space H there is a total orthonormal sequence $\tilde{F}$ which contains F.

9. Let M be a total set in an inner product space X. If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$, show that $v = w$.

10. Let M be a subset of a Hilbert space H, and let $v, w \in H$. Suppose that $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$, show that M is total in H.

3.7 Legendre, Hermite and Laguerre Polynomials

The theory of Hilbert spaces has applications to various solid topics in analysis. In the present section we discuss some total orthogonal and orthonormal sequences which are used quite frequently in connection with practical problems (for instance, in quantum mechanics, as we shall see in Chap. 11). Properties of these sequences have been investigated in great detail. A standard reference is A. Erdélyi et al. (1953–55) listed in Appendix 3.

The present section is optional.

3.7-1 Legendre polynomials. The inner product space X of all continuous real-valued functions on $[-1, 1]$ with inner product defined by

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)\, dt$$

can be completed according to Theorem 3.2-3. This gives a Hilbert space which is denoted by $L^2[-1, 1]$; cf. also Example 3.1-5.

We want to obtain a total orthonormal sequence in $L^2[-1, 1]$ which consists of functions that are easy to handle. Polynomials are of this type, and we shall succeed by a very simple idea. We start from the powers $x_0, x_1, x_2, \cdots$ where

$$(1) \qquad x_0(t)=1, \qquad x_1(t)=t, \quad \cdots, \quad x_j(t)=t^j, \cdots \qquad t\in[-1, 1].$$

This sequence is linearly independent. (Proof?) Applying the Gram-Schmidt process (Sec. 3.4), we obtain an orthonormal sequence (e_n). Each e_n is a polynomial since in the process we take linear combinations of the x_j's. The degree of e_n is n, as we shall see.

(e_n) is total in $L^2[-1, 1]$.

Proof. By Theorem 3.2-3 the set $W=A(X)$ is dense in $L^2[-1, 1]$. Hence for any fixed $x \in L^2[-1, 1]$ and given $\varepsilon>0$ there is a continuous function y defined on $[-1, 1]$ such that

$$\|x-y\|<\frac{\varepsilon}{2}.$$

For this y there is a polynomial z such that for all $t\in[-1, 1]$,

$$|y(t)-z(t)|<\frac{\varepsilon}{2\sqrt{2}}.$$

This follows from the Weierstrass approximation theorem to be proved in Sec. 4.11 and implies

$$\|y-z\|^2=\int_{-1}^{1}|y(t)-z(t)|^2\,dt<2\left(\frac{\varepsilon}{2\sqrt{2}}\right)^2=\frac{\varepsilon^2}{4}.$$

Together, by the triangle inequality,

$$\|x-z\|\leqq\|x-y\|+\|y-z\|<\varepsilon.$$

The definition of the Gram-Schmidt process shows that, by (1), we have $z\in\operatorname{span}\{e_0,\cdots,e_m\}$ for sufficiently large m. Since $x\in L^2[-1,1]$ and $\varepsilon>0$ were arbitrary, this proves totality of (e_n). ∎

For practical purposes one needs explicit formulas. We claim that

(2a)
$$e_n(t)=\sqrt{\frac{2n+1}{2}}\,P_n(t)\qquad\qquad n=0,1,\cdots$$

where

(2b)
$$P_n(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}[(t^2-1)^n].$$

P_n is called the *Legendre polynomial of order n.* Formula (2b) is called *Rodrigues' formula.* The square root in (2a) has the effect that $P_n(1)=1$, a property which we shall not prove since we do not need it.

By applying the binomial theorem to $(t^2-1)^n$ and differentiating the result n times term by term we obtain from (2b)

(2c)
$$P_n(t)=\sum_{j=0}^{N}(-1)^j\frac{(2n-2j)!}{2^n j!\,(n-j)!\,(n-2j)!}\,t^{n-2j}$$

where $N=n/2$ if n is even and $N=(n-1)/2$ if n is odd. Hence (Fig. 35)

(2*)
$$P_0(t)=1\qquad\qquad P_1(t)=t$$
$$P_2(t)=\tfrac{1}{2}(3t^2-1)\qquad\qquad P_3(t)=\tfrac{1}{2}(5t^3-3t)$$
$$P_4(t)=\tfrac{1}{8}(35t^4-30t^2+3)\qquad\qquad P_5(t)=\tfrac{1}{8}(63t^5-70t^3+15t)$$

etc.

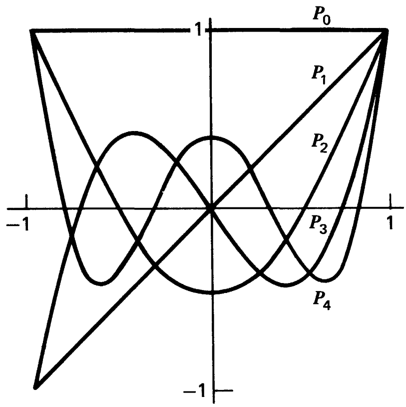

Fig. 35. Legendre polynomials

Proof of (2a) *and* (2b). In part (*a*) we show that (2b) implies

$$\|P_n\| = \left[\int_{-1}^{1} P_n^{\,2}(t)\,dt\right]^{1/2} = \sqrt{\frac{2}{2n+1}} \tag{3}$$

so that e_n in (2a) comes out with the correct norm, which is 1. In part (*b*) we prove that (P_n) is an orthogonal sequence in the space $L^2[-1, 1]$. This suffices to establish (2a) and (2b) for the following reason. We denote the right-hand side of (2a) at first by $y_n(t)$. Then y_n is a polynomial of degree n, and those parts (*a*) and (*b*) imply that (y_n) is an orthonormal sequence in $L^2[-1, 1]$. Let

$$Y_n = \operatorname{span}\{e_0, \cdots, e_n\} = \operatorname{span}\{x_0, \cdots, x_n\} = \operatorname{span}\{y_0, \cdots, y_n\};$$

here the second equality sign follows from the algorithm of the Gram–Schmidt process and the last equality sign from $\dim Y_n = n+1$ together with the linear independence of $\{y_0, \cdots, y_n\}$ stated in 3.4-2. Hence y_n has a representation

$$y_n = \sum_{j=0}^{n} \alpha_j e_j. \tag{4}$$

Now by the orthogonality,

$$y_n \perp Y_{n-1} = \operatorname{span}\{y_0, \cdots, y_{n-1}\} = \operatorname{span}\{e_0, \cdots, e_{n-1}\}.$$

This implies that for $k=0, \cdots, n-1$ we have

$$0=\langle y_n, e_k\rangle=\sum_{j=0}^{n} \alpha_j\langle e_j, e_k\rangle=\alpha_k.$$

Hence (4) reduces to $y_n=\alpha_n e_n$. Here $|\alpha_n|=1$ since $\|y_n\|=\|e_n\|=1$. Actually, $\alpha_n=+1$ or -1 since both y_n and e_n are real. Now $y_n(t)>0$ for sufficiently large t since the coefficient of t^n in (2c) is positive. Also $e_n(t)>0$ for sufficiently large t, as can be seen from $x_n(t)=t^n$ and (13) and (14) in Sec. 3.4. Hence $\alpha=+1$ and $y_n=e_n$, which establishes (2a) with P_n given by (2b).

This altogether shows that after the presentation of the aforementioned parts (*a*) and (*b*) the proof will be complete.

(a) We derive (3) from (2b). We write $u=t^2-1$. The function u^n and its derivatives $(u^n)', \cdots, (u^n)^{(n-1)}$ are zero at $t=\pm 1$, and $(u^n)^{(2n)}=(2n)!$. Integrating n times by parts, we thus obtain from (2b)

$$\begin{aligned}
(2^n n!)^2\|P_n\|^2 &= \int_{-1}^{1} (u^n)^{(n)}(u^n)^{(n)}\,dt \\
&= (u^n)^{(n-1)}(u^n)^{(n)}\Big|_{-1}^{1} - \int_{-1}^{1} (u^n)^{(n-1)}(u^n)^{(n+1)}\,dt \\
&= \cdots \\
&= (-1)^n(2n)!\int_{-1}^{1} u^n\,dt \\
&= 2(2n)!\int_0^1 (1-t^2)^n\,dt \\
&= 2(2n)!\int_0^{\pi/2} \cos^{2n+1}\tau\,d\tau \qquad (t=\sin\tau) \\
&= \frac{2^{2n+1}(n!)^2}{2n+1}.
\end{aligned}$$

Division by $(2^n n!)^2$ yields (3).

(b) We show that $\langle P_m, P_n\rangle=0$ where $0\leqq m<n$. Since P_m is a polynomial, it suffices to prove that $\langle x_m, P_n\rangle=0$ for $m<n$, where

x_m is defined by (1). This result is obtained by m integrations by parts:

$$\begin{aligned} 2^n n!\,\langle x_m, P_n\rangle &= \int_{-1}^{1} t^m (u^n)^{(n)}\,dt \\ &= t^m (u^n)^{(n-1)}\Big|_{-1}^{1} - m\int_{-1}^{1} t^{m-1}(u^n)^{(n-1)}\,dt \\ &= \cdots \\ &= (-1)^m m! \int_{-1}^{1} (u^n)^{(n-m)}\,dt \\ &= (-1)^m m!\,(u^n)^{(n-m-1)}\Big|_{-1}^{1} = 0. \end{aligned}$$

This completes the proof of (2a) and (2b). ∎

The Legendre polynomials are solutions of the important *Legendre differential equation*

(5) $$(1-t^2)P_n'' - 2tP_n' + n(n+1)P_n = 0,$$

and (2c) can also be obtained by applying the power series method to (5).

Furthermore, a total orthonormal sequence in the space $L^2[a, b]$ is (q_n), where

(6) $$q_n = \frac{1}{\|p_n\|}\,p_n, \qquad p_n(t) = P_n(s), \qquad s = 1 + 2\,\frac{t-b}{b-a}.$$

The proof follows if we note that $a \leqq t \leqq b$ corresponds to $-1 \leqq s \leqq 1$ and the orthogonality is preserved under this linear transformation $t \longmapsto s$.

We thus have a total orthonormal sequence in $L^2[a, b]$ for any compact interval $[a, b]$. Theorem 3.6-4 thus implies:

The real space $L^2[a, b]$ is separable.

3.7-2 Hermite polynomials. Further spaces of practical interest are $L^2(-\infty, +\infty)$, $L^2[a, +\infty)$ and $L^2(-\infty, b]$. These are not taken care of by 3.7-1. Since the intervals of integration are infinite, the powers

$x_0, x_1, \cdots$ in 3.7-1 alone would not help. But if we multiply each of them by a simple function which decreases sufficiently rapidly, we can hope to obtain integrals which are finite. Exponential functions with a suitable exponent seem to be a natural choice.

We consider the real space $L^2(-\infty, +\infty)$. The inner product is given by

$$\langle x, y\rangle = \int_{-\infty}^{+\infty} x(t)y(t)\,dt.$$

We apply the Gram-Schmidt process to the sequence of functions defined by

$$w(t) = e^{-t^2/2}, \qquad tw(t), \qquad t^2 w(t), \cdots.$$

The factor 1/2 in the exponent is purely conventional and has no deeper meaning. These functions are elements of $L^2(-\infty, +\infty)$. In fact, they are bounded on $\mathbf{R}$, say, $|t^n w(t)| \leqq k_n$ for all t; thus,

$$\left|\int_{-\infty}^{+\infty} t^m e^{-t^2/2}\, t^n e^{-t^2/2}\,dt\right| \leqq k_{m+n} \int_{-\infty}^{+\infty} e^{-t^2/2}\,dt = k_{m+n}\sqrt{2\pi}.$$

The Gram-Schmidt process gives the orthonormal sequence (e_n), where (Fig. 36)

(7a)
$$e_n(t) = \frac{1}{(2^n n!\,\sqrt{\pi})^{1/2}}\, e^{-t^2/2} H_n(t)$$

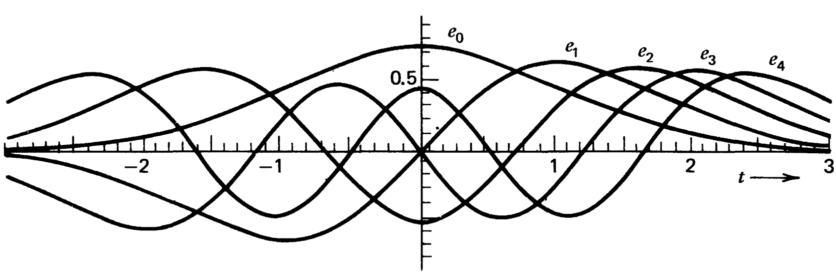

Fig. 36. Functions e_n in (7a) involving Hermite polynomials

and

(7b) $$H_0(t)=1, \qquad H_n(t)=(-1)^n e^{t^2}\frac{d^n}{dt^n}(e^{-t^2}) \qquad n=1,2,\cdots.$$

H_n is called the *Hermite polynomial of order n.*

Performing the differentiations indicated in (7b), we obtain

(7c) $$H_n(t)=n!\sum_{j=0}^{N}(-1)^j\frac{2^{n-2j}}{j!\,(n-2j)!}\,t^{n-2j}$$

where $N=n/2$ if n is even and $N=(n-1)/2$ if n is odd. Note that this can also be written

(7c′) $$H_n(t)=\sum_{j=0}^{N}\frac{(-1)^j}{j!}\,n(n-1)\cdots(n-2j+1)(2t)^{n-2j}.$$

Explicit expressions for the first few Hermite polynomials are

(7*)
$$H_0(t)=1 \qquad H_1(t)=2t$$
$$H_2(t)=4t^2-2 \qquad H_3(t)=8t^3-12t$$
$$H_4(t)=16t^4-48t^2+12 \qquad H_5(t)=32t^5-160t^3+120t.$$

The sequence (e_n) *defined by* (7a) *and* (7b) *is orthonormal.*

Proof. (7a) and (7b) show that we must prove

(8) $$\int_{-\infty}^{+\infty} e^{-t^2}H_m(t)H_n(t)\,dt=\begin{cases}0 & \text{if } m\neq n\\ 2^n n!\,\sqrt{\pi} & \text{if } m=n.\end{cases}$$

Differentiating (7c′), we obtain for $n\geqq 1$

$$H_n'(t)=2n\sum_{j=0}^{M}\frac{(-1)^j}{j!}(n-1)(n-2)\cdots(n-2j)(2t)^{n-1-2j}$$
$$=2nH_{n-1}(t)$$

where $M=(n-2)/2$ if n is even and $M=(n-1)/2$ if n is odd. We apply this formula to H_m, assume $m\leqq n$, denote the exponential function in (8) by v, for simplicity, and integrate m times by parts. Then, by (7b),

$$\begin{aligned}(-1)^n \int_{-\infty}^{+\infty} e^{-t^2} H_m(t)H_n(t)\,dt &= \int_{-\infty}^{+\infty} H_m(t)v^{(n)}\,dt \\ &= H_m(t)v^{(n-1)}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2mH_{m-1}(t)v^{(n-1)}\,dt \\ &= -2m\int_{-\infty}^{+\infty} H_{m-1}(t)v^{(n-1)}\,dt \\ &= \cdots \\ &= (-1)^m 2^m m! \int_{-\infty}^{+\infty} H_0(t)v^{(n-m)}\,dt.\end{aligned}$$

Here $H_0(t)=1$. If $m<n$, integrating once more, we obtain 0 since v and its derivatives approach zero as $t \longrightarrow +\infty$ or $t \longrightarrow -\infty$. This proves orthogonality of (e_n). We prove (8) for $m=n$, which entails $\|e_n\|=1$ by (7a). If $m=n$, for the last integral, call it J, we obtain

$$J = \int_{-\infty}^{+\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

This is a familiar result. To verify it, consider J^2, use polar coordinates r, θ and $ds\,dt = r\,dr\,d\theta$, finding

$$\begin{aligned}J^2 = \int_{-\infty}^{+\infty} e^{-s^2}\,ds \int_{-\infty}^{+\infty} e^{-t^2}\,dt &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(s^2+t^2)}\,ds\,dt \\ &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta \\ &= 2\pi \cdot \tfrac{1}{2} = \pi.\end{aligned}$$

This proves (8), hence the orthonormality of (e_n). ∎

Classically speaking, one often expresses (8) by saying that the H_n's form an orthogonal sequence with respect to the *weight function* w^2, where w is the function defined at the beginning.

It can be shown that (e_n) *defined by* (7a), (7b) *is total in the real space* $L^2(-\infty, +\infty)$. *Hence this space is separable.* (Cf. 3.6-4.)

We finally mention that the Hermite polynomials H_n satisfy the *Hermite differential equation*

$$H_n'' - 2tH_n' + 2nH_n = 0. \tag{9}$$

Warning. Unfortunately, the terminology in the literature is not unique. In fact, the functions He_n defined by

$$He_0(t) = 1, \qquad He_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n}(e^{-t^2/2}) \qquad n = 1, 2, \cdots$$

are also called "Hermite polynomials" and, to make things worse, are sometimes denoted by H_n.

An application of Hermite polynomials in quantum mechanics will be considered in Sec. 11.3.

3.7-3 Laguerre polynomials. A total orthonormal sequence in $L^2(-\infty, b]$ or $L^2[a, +\infty)$ can be obtained from such a sequence in $L^2[0, +\infty)$ by the transformations $t = b - s$ and $t = s + a$, respectively.

We consider $L^2[0, +\infty)$. Applying the Gram-Schmidt process to the sequence defined by

$$e^{-t/2}, \qquad te^{-t/2}, \qquad t^2 e^{-t/2}, \qquad \cdots$$

we obtain an orthonormal sequence (e_n). It can be shown that (e_n) is total in $L^2[0, +\infty)$ and is given by (Fig. 37)

$$e_n(t) = e^{-t/2} L_n(t) \qquad n = 0, 1, \cdots \tag{10a}$$

where the *Laguerre polynomial of order n* is defined by

$$L_0(t) = 1, \qquad L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n}(t^n e^{-t}) \qquad n = 1, 2, \cdots, \tag{10b}$$

that is,

$$L_n(t) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} t^j. \tag{10c}$$

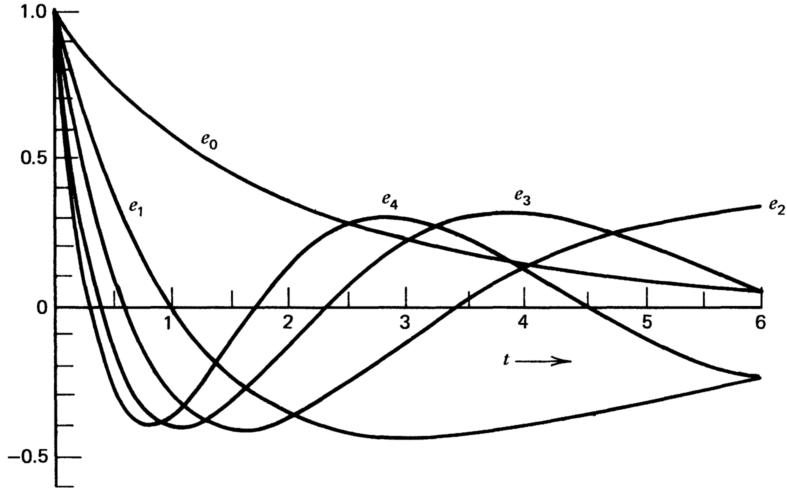

Fig. 37. Functions e_n in (10a) involving Laguerre polynomials

Explicit expressions for the first few Laguerre polynomials are

$$L_0(t)=1 \qquad L_1(t)=1-t$$

(10*) $$L_2(t)=1-2t+\tfrac{1}{2}t^2 \qquad L_3(t)=1-3t+\tfrac{3}{2}t^2-\tfrac{1}{6}t^3$$

$$L_4(t)=1-4t+3t^2-\tfrac{2}{3}t^3+\tfrac{1}{24}t^4.$$

The Laguerre polynomials L_n are solutions of the *Laguerre differential equation*

(11) $$tL_n''+(1-t)L_n'+nL_n=0.$$

For further details, see A. Erdélyi et al. (1953–55); cf. also R. Courant and D. Hilbert (1953–62), vol. I.

Problems

1. Show that the Legendre differential equation can be written

$$[(1-t^2)P_n']'=-n(n+1)P_n.$$

Multiply this by P_m. Multiply the corresponding equation for P_m by

$-P_n$ and add the two equations. Integrating the resulting equation from -1 to 1, show that (P_n) is an orthogonal sequence in the space $L^2[-1, 1]$.

2. Derive (2c) from (2b).

3. (Generating function) Show that

$$\frac{1}{\sqrt{1-2tw+w^2}} = \sum_{n=0}^{\infty} P_n(t)w^n.$$

The function on the left is called a *generating function* of the Legendre polynomials. Generating functions are useful in connection with various special functions; cf. R. Courant and D. Hilbert (1953–62), A. Erdélyi et al. (1953–55).

4. Show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

where r is the distance between given points A_1 and A_2 in $\mathbf{R}^3$, as shown in Fig. 38, and $r_2 > 0$. (This formula is useful in potential theory.)

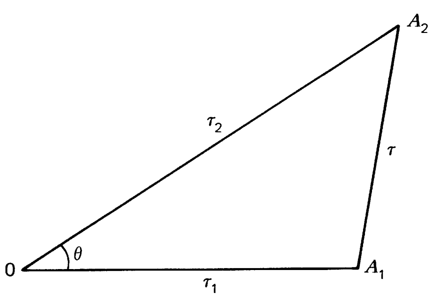

Fig. 38. Problem 4

5. Obtain the Legendre polynomials by the power series method as follows. Substitute $x(t) = c_0 + c_1t + c_2t^2 + \cdots$ into Legendre's equation and show that by determining the coefficients one obtains the solution $x = c_0x_1 + c_1x_2$, where

$$x_1(t) = 1 - \frac{n(n+1)}{2!}t^2 + \frac{(n-2)n(n+1)(n+3)}{4!}t^4 - + \cdots$$

and

$$x_2 = t - \frac{(n-1)(n+2)}{3!} t^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} t^5 - + \cdots.$$

Show that for $n \in \mathbf{N}$, one of these two functions reduces to a polynomial, which agrees with P_n if one chooses $c_n = (2n)!/2^n(n!)^2$ as the coefficient of t^n.

6. (Generating function) Show that

$$\exp(2wt - w^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(t) w^n.$$

The function on the left is called a *generating function* of the Hermite polynomials.

7. Using (7b), show that

$$H_{n+1}(t) = 2tH_n(t) - H_n'(t).$$

8. Differentiating the generating function in Prob. 6 with respect to t, show that

$$H_n'(t) = 2nH_{n-1}(t) \qquad (n \geqq 1)$$

and, using Prob. 7, show that H_n satisfies the Hermite differential equation.

9. Solve the differential equation $y'' + (2n+1-t^2)y = 0$ in terms of Hermite polynomials.

10. Using Prob. 8, show that

$$(e^{-t^2} H_n')' = -2ne^{-t^2} H_n.$$

Using this and the method explained in Prob. 1, show that the functions defined by (7a) are orthogonal on $\mathbf{R}$.

11. (Generating function) Using (10c), show that

$$\psi(t, w) = \frac{1}{1-w} \exp\left[-\frac{tw}{1-w}\right] = \sum_{n=0}^{\infty} L_n(t) w^n.$$

12. Differentiating ψ in Prob. 11 with respect to w, show that

(a) $$(n+1)L_{n+1}(t)-(2n+1-t)L_n(t)+nL_{n-1}(t)=0.$$

Differentiating ψ with respect to t, show that

(b) $$L_{n-1}(t)=L'_{n-1}(t)-L'_n(t).$$

13. Using Prob. 12, show that

(c) $$tL'_n(t)=nL_n(t)-nL_{n-1}(t).$$

Using this and (b) in Prob. 12, show that L_n satisfies Laguerre's differential equation (11).

14. Show that the functions in (10a) have norm 1.

15. Show that the functions in (10a) constitute an orthogonal sequence in the space $L^2[0, +\infty)$.

3.8 Representation of Functionals on Hilbert Spaces

It is of practical importance to know the general form of bounded linear functionals on various spaces. This was pointed out and explained in Sec. 2.10. For general Banach spaces such formulas and their derivation can sometimes be complicated. However, for a Hilbert space the situation is surprisingly simple:

3.8-1 Riesz's Theorem (Functionals on Hilbert spaces). *Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely,*

(1) $$f(x)=\langle x, z\rangle$$

where z depends on f, is uniquely determined by f and has norm

(2) $$\|z\|=\|f\|.$$

Proof. We prove that
(*a*) f has a representation (1),
(*b*) z in (1) is unique,
(*c*) formula (2) holds.
The details are as follows.

(a) If $f=0$, then (1) and (2) hold if we take $z=0$. Let $f\neq 0$. To motivate the idea of the proof, let us ask what properties z must have if a representation (1) exists. First of all, $z\neq 0$ since otherwise $f=0$. Second, $\langle x, z\rangle = 0$ for all x for which $f(x)=0$, that is, for all x in the null space $\mathcal{N}(f)$ of f. Hence $z \perp \mathcal{N}(f)$. This suggests that we consider $\mathcal{N}(f)$ and its orthogonal complement $\mathcal{N}(f)^{\perp}$.

$\mathcal{N}(f)$ is a vector space by 2.6-9 and is closed by 2.7-10. Furthermore, $f\neq 0$ implies $\mathcal{N}(f)\neq H$, so that $\mathcal{N}(f)^{\perp}\neq\{0\}$ by the projection theorem 3.3-4. Hence $\mathcal{N}(f)^{\perp}$ contains a $z_0\neq 0$. We set

$$v = f(x)z_0 - f(z_0)x$$

where $x\in H$ is arbitrary. Applying f, we obtain

$$f(v) = f(x)f(z_0) - f(z_0)f(x) = 0.$$

This show that $v\in\mathcal{N}(f)$. Since $z_0 \perp \mathcal{N}(f)$, we have

$$\begin{aligned} 0 = \langle v, z_0\rangle &= \langle f(x)z_0 - f(z_0)x, z_0\rangle \\ &= f(x)\langle z_0, z_0\rangle - f(z_0)\langle x, z_0\rangle. \end{aligned}$$

Noting that $\langle z_0, z_0\rangle = \|z_0\|^2 \neq 0$, we can solve for $f(x)$. The result is

$$f(x) = \frac{f(z_0)}{\langle z_0, z_0\rangle}\langle x, z_0\rangle.$$

This can be written in the form (1), where

$$z = \frac{\overline{f(z_0)}}{\langle z_0, z_0\rangle} z_0.$$

Since $x\in H$ was arbitrary, (1) is proved.

(b) We prove that z in (1) is unique. Suppose that for all $x \in H$,

$$f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle.$$

Then $\langle x, z_1 - z_2 \rangle = 0$ for all x. Choosing the particular $x = z_1 - z_2$, we have

$$\langle x, z_1 - z_2 \rangle = \langle z_1 - z_2, z_1 - z_2 \rangle = \|z_1 - z_2\|^2 = 0.$$

Hence $z_1 - z_2 = 0$, so that $z_1 = z_2$, the uniqueness.

(c) We finally prove (2). If $f = 0$, then $z = 0$ and (2) holds. Let $f \neq 0$. Then $z \neq 0$. From (1) with $x = z$ and (3) in Sec. 2.8 we obtain

$$\|z\|^2 = \langle z, z \rangle = f(z) \leqq \|f\| \, \|z\|.$$

Division by $\|z\| \neq 0$ yields $\|z\| \leqq \|f\|$. It remains to show that $\|f\| \leqq \|z\|$. From (1) and the Schwarz inequality (Sec. 3.2) we see that

$$|f(x)| = |\langle x, z \rangle| \leqq \|x\| \, \|z\|.$$

This implies

$$\|f\| = \sup_{\|x\|=1} |\langle x, z \rangle| \leqq \|z\|. \qquad \blacksquare$$

The idea of the uniqueness proof in part (*b*) is worth noting for later use:

3.8-2 Lemma (Equality). *If $\langle v_1, w \rangle = \langle v_2, w \rangle$ for all w in an inner product space X, then $v_1 = v_2$. In particular, $\langle v_1, w \rangle = 0$ for all $w \in X$ implies $v_1 = 0$.*

Proof. By assumption, for all w,

$$\langle v_1 - v_2, w \rangle = \langle v_1, w \rangle - \langle v_2, w \rangle = 0.$$

For $w = v_1 - v_2$ this gives $\|v_1 - v_2\|^2 = 0$. Hence $v_1 - v_2 = 0$, so that $v_1 = v_2$. In particular, $\langle v_1, w \rangle = 0$ with $w = v_1$ gives $\|v_1\|^2 = 0$, so that $v_1 = 0$. $\blacksquare$

The practical usefulness of bounded linear functionals on Hilbert spaces results to a large extent from the simplicity of the Riesz representation (1).

Furthermore, (1) is quite important in the theory of operators on Hilbert spaces. In particular, this refers to the Hilbert-adjoint operator T^* of a bounded linear operator T which we shall define in the next section. For this purpose we need a preparation which is of general interest, too. We begin with the following definition.

3.8-3 Definition (Sesquilinear form). Let X and Y be vector spaces over the same field K ($=\mathbf{R}$ or $\mathbf{C}$). Then a *sesquilinear form* (or *sesquilinear functional*) h on $X \times Y$ is a mapping

$$h\colon X \times Y \longrightarrow K$$

such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and all scalars α, β,

(3)

(a) $$h(x_1+x_2, y) = h(x_1, y)+h(x_2, y)$$

(b) $$h(x, y_1+y_2) = h(x, y_1)+h(x, y_2)$$

(c) $$h(\alpha x, y) = \alpha h(x, y)$$

(d) $$h(x, \beta y) = \bar{\beta} h(x, y).$$

Hence h is *linear* in the first argument and *conjugate linear* in the second one. If X and Y are real ($K=\mathbf{R}$), then (3d) is simply

$$h(x, \beta y) = \beta h(x, y)$$

and h is called *bilinear* since it is linear in both arguments.

If X and Y are normed spaces and if there is a real number c such that for all x, y

(4) $$|h(x, y)| \leqq c \|x\| \|y\|,$$

then h is said to be *bounded*, and the number

(5) $$\|h\| = \sup_{\substack{x \in X-\{0\} \\ y \in Y-\{0\}}} \frac{|h(x, y)|}{\|x\| \|y\|} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x, y)|$$

is called the *norm* of h. ∎

For example, the inner product is sesquilinear and bounded.

Note that from (4) and (5) we have

$$|h(x, y)| \leqq \|h\| \, \|x\| \, \|y\|. \tag{6}$$

The term "sesquilinear" was motivated in Sec. 3.1. In Def. 3.8-3, both words "form" and "functional" are common, the usage of one or the other being largely a matter of individual taste. Perhaps it is slightly preferable to use "form" in this two-variable case and reserve the word "functional" to one-variable cases such as that in Theorem 3.8-1. This is what we shall do.

It is quite interesting that from Theorem 3.8-1 we can obtain a general representation of sesquilinear forms on Hilbert spaces as follows.

3.8-4 Theorem (Riesz representation). *Let H_1, H_2 be Hilbert spaces and*

$$h: H_1 \times H_2 \longrightarrow K$$

a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle Sx, y \rangle \tag{7}$$

where $S: H_1 \longrightarrow H_2$ is a bounded linear operator. S is uniquely determined by h and has norm

$$\|S\| = \|h\|. \tag{8}$$

Proof. We consider $\overline{h(x, y)}$. This is linear in y, because of the bar. To make Theorem 3.8-1 applicable, we keep x fixed. Then that theorem yields a representation in which y is variable, say,

$$\overline{h(x, y)} = \langle y, z \rangle.$$

Hence

$$h(x, y) = \langle z, y \rangle. \tag{9}$$

Here $z \in H_2$ is unique but, of course, depends on our fixed $x \in H_1$. It follows that (9) with variable x defines an operator

$$S\colon H_1 \longrightarrow H_2 \qquad \text{given by} \qquad z = Sx.$$

Substituting $z = Sx$ in (9), we have (7).

S is linear. In fact, its domain is the vector space H_1, and from (7) and the sesquilinearity we obtain

$$\begin{aligned}\langle S(\alpha x_1 + \beta x_2), y\rangle &= h(\alpha x_1 + \beta x_2, y) \\ &= \alpha h(x_1, y) + \beta h(x_2, y) \\ &= \alpha\langle Sx_1, y\rangle + \beta\langle Sx_2, y\rangle \\ &= \langle \alpha Sx_1 + \beta Sx_2, y\rangle\end{aligned}$$

for all y in H_2, so that by Lemma 3.8-2,

$$S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2.$$

S is bounded. Indeed, leaving aside the trivial case $S = 0$, we have from (5) and (7)

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\,\|y\|} \geqq \sup_{\substack{x \neq 0 \\ Sx \neq 0}} \frac{|\langle Sx, Sx\rangle|}{\|x\|\,\|Sx\|} = \sup_{x \neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|.$$

This proves boundedness. Moreover, $\|h\| \geqq \|S\|$.

We now obtain (8) by noting that $\|h\| \leqq \|S\|$ follows by an application of the Schwarz inequality:

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\,\|y\|} \leqq \sup_{x \neq 0} \frac{\|Sx\|\,\|y\|}{\|x\|\,\|y\|} = \|S\|.$$

S is unique. In fact, assuming that there is a linear operator $T\colon H_1 \longrightarrow H_2$ such that for all $x \in H_1$ and $y \in H_2$ we have

$$h(x, y) = \langle Sx, y\rangle = \langle Tx, y\rangle,$$

we see that $Sx = Tx$ by Lemma 3.8-2 for all $x \in H_1$. Hence $S = T$ by definition. ∎

Problems

1. **(Space $\mathbf{R}^3$)** Show that any linear functional f on $\mathbf{R}^3$ can be represented by a dot product:

$$f(x) = x \cdot z = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3.$$

2. **(Space l^2)** Show that every bounded linear functional f on l^2 can be represented in the form

$$f(x) = \sum_{j=1}^{\infty} \xi_j \bar{\zeta}_j \qquad [z = (\zeta_j) \in l^2].$$

3. If z is any fixed element of an inner product space X, show that $f(x) = \langle x, z \rangle$ defines a bounded linear functional f on X, of norm $\|z\|$.

4. Consider Prob. 3. If the mapping $X \longrightarrow X'$ given by $z \longmapsto f$ is surjective, show that X must be a Hilbert space.

5. Show that the dual space of the real space l^2 is l^2. (Use 3.8-1.)

6. Show that Theorem 3.8-1 defines an isometric bijection $T\colon H \longrightarrow H'$, $z \longmapsto f_z = \langle \cdot, z \rangle$ which is not linear but *conjugate linear,* that is, $\alpha z + \beta v \longmapsto \bar{\alpha} f_z + \bar{\beta} f_v$.

7. Show that the dual space H' of a Hilbert space H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$ defined by

$$\langle f_z, f_v \rangle_1 = \overline{\langle z, v \rangle} = \langle v, z \rangle,$$

where $f_z(x) = \langle x, z \rangle$, etc.

8. Show that any Hilbert space H is isomorphic (cf. Sec. 3.6) with its second dual space $H'' = (H')'$. (This property is called *reflexivity* of H. It will be considered in more detail for normed spaces in Sec. 4.6.)

9. **(Annihilator)** Explain the relation between M^a in Prob. 13, Sec. 2.10, and $M^\perp$ in Sec. 3.3 in the case of a subset $M \neq \varnothing$ of a Hilbert space H.

10. Show that the inner product $\langle \cdot, \cdot \rangle$ on an inner product space X is a bounded sesquilinear form h. What is $\|h\|$ in this case?

11. If X is a vector space and h a sesquilinear form on $X \times X$, show that $f_1(x) = h(x, y_0)$ with fixed y_0 defines a linear functional f_1 on X, and so does $f_2(y) = \overline{h(x_0, y)}$ with fixed x_0.

12. Let X and Y be normed spaces. Show that a bounded sesquilinear form h on $X \times Y$ is jointly continuous in both variables.

13. (Hermitian form) Let X be a vector space over a field K. A *Hermitian sesquilinear form* or, simply, *Hermitian form* h on $X \times X$ is a mapping $h\colon X \times X \longrightarrow K$ such that for all $x, y, z \in X$ and $\alpha \in K$,

$$h(x+y, z) = h(x, z) + h(y, z)$$

$$h(\alpha x, y) = \alpha h(x, y)$$

$$h(x, y) = \overline{h(y, x)}.$$

What is the last condition if $K = \mathbf{R}$? What condition must be added for h to be an inner product on X?

14. (Schwarz inequality) Let X be a vector space and h a Hermitian form on $X \times X$. This form is said to be *positive semidefinite* if $h(x, x) \geqq 0$ for all $x \in X$. Show that then h satisfies the *Schwarz inequality*

$$|h(x, y)|^2 \leqq h(x, x)h(y, y).$$

15. (Seminorm) If h satisfies the conditions in Prob. 14, show that

$$p(x) = \sqrt{h(x, x)} \qquad (\geqq 0)$$

defines a seminorm on X. (Cf. Prob. 12, Sec. 2.3.)

3.9 Hilbert-Adjoint Operator

The results of the previous section will now enable us to introduce the Hilbert-adjoint operator of a bounded linear operator on a Hilbert space. This operator was suggested by problems in matrices and linear differential and integral equations. We shall see that it also helps to define three important classes of operators (called *self-adjoint*, *unitary* and *normal operators*) which have been studied extensively because they play a key role in various applications.

3.9-1 Definition (Hilbert-adjoint operator T^*). Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the *Hilbert-adjoint operator* T^* of T is the operator

$$T^*: H_2 \longrightarrow H_1$$

such that[5] for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle. \tag{1}$$

∎

Of course, we should first show that this definition makes sense, that is, we should prove that for a given T such a T^* does exist:

3.9-2 Theorem (Existence). *The Hilbert-adjoint operator T^* of T in Def.* 3.9-1 *exists, is unique and is a bounded linear operator with norm*

$$\|T^*\| = \|T\|. \tag{2}$$

Proof. The formula

$$h(y, x) = \langle y, Tx\rangle \tag{3}$$

defines a sesquilinear form on $H_2 \times H_1$ because the inner product is sesquilinear and T is linear. In fact, conjugate linearity of the form is seen from

$$\begin{aligned} h(y, \alpha x_1 + \beta x_2) &= \langle y, T(\alpha x_1 + \beta x_2)\rangle \\ &= \langle y, \alpha Tx_1 + \beta Tx_2\rangle \\ &= \bar{\alpha}\langle y, Tx_1\rangle + \bar{\beta}\langle y, Tx_2\rangle \\ &= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2). \end{aligned}$$

h is bounded. Indeed, by the Schwarz inequality,

$$|h(y, x)| = |\langle y, Tx\rangle| \leqq \|y\| \, \|Tx\| \leqq \|T\| \, \|x\| \, \|y\|.$$

[5] We may denote inner products on H_1 and H_2 by the same symbol since the factors show to which space an inner product refers.

This also implies $\|h\| \leqq \|T\|$. Moreover we have $\|h\| \geqq \|T\|$ from

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle y, Tx\rangle|}{\|y\| \, \|x\|} \geqq \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{|\langle Tx, Tx\rangle|}{\|Tx\| \, \|x\|} = \|T\|.$$

Together,

$$\|h\| = \|T\|. \tag{4}$$

Theorem 3.8-4 gives a Riesz representation for h; writing T^* for S, we have

$$h(y, x) = \langle T^* y, x\rangle, \tag{5}$$

and we know from that theorem that $T^*: H_2 \longrightarrow H_1$ is a uniquely determined bounded linear operator with norm [cf. (4)]

$$\|T^*\| = \|h\| = \|T\|.$$

This proves (2). Also $\langle y, Tx\rangle = \langle T^* y, x\rangle$ by comparing (3) and (5), so that we have (1) by taking conjugates, and we now see that T^* is in fact the operator we are looking for. ▮

In our study of properties of Hilbert-adjoint operators it will be convenient to make use of the following lemma.

3.9-3 Lemma (Zero operator). *Let X and Y be inner product spaces and $Q: X \longrightarrow Y$ a bounded linear operator. Then:*

(a) *$Q = 0$ if and only if $\langle Qx, y\rangle = 0$ for all $x \in X$ and $y \in Y$.*

(b) *If $Q: X \longrightarrow X$, where X is complex, and $\langle Qx, x\rangle = 0$ for all $x \in X$, then $Q = 0$.*

Proof. **(a)** $Q = 0$ means $Qx = 0$ for all x and implies

$$\langle Qx, y\rangle = \langle 0, y\rangle = 0\langle w, y\rangle = 0.$$

Conversely, $\langle Qx, y\rangle = 0$ for all x and y implies $Qx = 0$ for all x by 3.8-2, so that $Q = 0$ by definition.

(b) By assumption, $\langle Qv, v\rangle = 0$ for every $v = \alpha x + y \in X$, that is,

$$\begin{aligned} 0 &= \langle Q(\alpha x + y), \alpha x + y \rangle \\ &= |\alpha|^2 \langle Qx, x \rangle + \langle Qy, y \rangle + \alpha \langle Qx, y \rangle + \bar{\alpha} \langle Qy, x \rangle. \end{aligned}$$

The first two terms on the right are zero by assumption. $\alpha = 1$ gives

$$\langle Qx, y \rangle + \langle Qy, x \rangle = 0.$$

$\alpha = i$ gives $\bar{\alpha} = -i$ and

$$\langle Qx, y \rangle - \langle Qy, x \rangle = 0.$$

By addition, $\langle Qx, y \rangle = 0$, and $Q = 0$ follows from (a). ∎

In part (b) of this lemma, it is essential that X be complex. Indeed, the conclusion may not hold if X is real. A counterexample is a rotation Q of the plane $\mathbf{R}^2$ through a right angle. Q is linear, and $Qx \perp x$, hence $\langle Qx, x \rangle = 0$ for all $x \in \mathbf{R}^2$, but $Q \neq 0$. (What about such a rotation in the complex plane?)

We can now list and prove some general properties of Hilbert-adjoint operators which one uses quite frequently in applying these operators.

3.9-4 Theorem (Properties of Hilbert-adjoint operators). *Let H_1, H_2 be Hilbert spaces, $S: H_1 \longrightarrow H_2$ and $T: H_1 \longrightarrow H_2$ bounded linear operators and α any scalar. Then we have*

(a) $\langle T^* y, x \rangle = \langle y, Tx \rangle \qquad (x \in H_1, y \in H_2)$

(b) $(S + T)^* = S^* + T^*$

(c) $(\alpha T)^* = \bar{\alpha} T^*$

(6) (d) $(T^*)^* = T$

(e) $\|T^* T\| = \|TT^*\| = \|T\|^2$

(f) $T^* T = 0 \iff T = 0$

(g) $(ST)^* = T^* S^* \qquad$ (assuming $H_2 = H_1$).

Proof. (*a*) From (1) we have (6a):

$$\langle T^*y, x\rangle = \overline{\langle x, T^*y\rangle} = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle.$$

(*b*) By (1), for all x and y,

$$\begin{aligned}\langle x, (S+T)^*y\rangle &= \langle (S+T)x, y\rangle \\ &= \langle Sx, y\rangle + \langle Tx, y\rangle \\ &= \langle x, S^*y\rangle + \langle x, T^*y\rangle \\ &= \langle x, (S^*+T^*)y\rangle.\end{aligned}$$

Hence $(S+T)^*y = (S^*+T^*)y$ for all y by 3.8-2, which is (6b) by definition.

(*c*) Formula (6c) must not be confused with the formula $T^*(\alpha x) = \alpha T^*x$. It is obtained from the following calculation and subsequent application of Lemma 3.9-3(*a*) to $Q = (\alpha T)^* - \bar{\alpha}T^*$.

$$\begin{aligned}\langle (\alpha T)^*y, x\rangle &= \langle y, (\alpha T)x\rangle \\ &= \langle y, \alpha(Tx)\rangle \\ &= \bar{\alpha}\langle y, Tx\rangle \\ &= \bar{\alpha}\langle T^*y, x\rangle \\ &= \langle \bar{\alpha}T^*y, x\rangle.\end{aligned}$$

(*d*) $(T^*)^*$ is written T^{**} and equals T since for all $x \in H_1$ and $y \in H_2$ we have from (6a) and (1)

$$\langle (T^*)^*x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle$$

and (6d) follows from Lemma 3.9-3(a) with $Q = (T^*)^* - T$.

(*e*) We see that $T^*T\colon H_1 \longrightarrow H_1$, but $TT^*\colon H_2 \longrightarrow H_2$. By the Schwarz inequality,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leqq \|T^*Tx\|\,\|x\| \leqq \|T^*T\|\,\|x\|^2.$$

Taking the supremum over all x of norm 1, we obtain $\|T\|^2 \leqq \|T^*T\|$. Applying (7), Sec. 2.7, and (2), we thus have

$$\|T\|^2 \leqq \|T^*T\| \leqq \|T^*\| \, \|T\| = \|T\|^2.$$

Hence $\|T^*T\| = \|T\|^2$. Replacing T by T^* and using again (2), we also have

$$\|T^{**}T^*\| = \|T^*\|^2 = \|T\|^2.$$

Here $T^{**} = T$ by (6d), so that (6e) is proved.

(*f*) From (6e) we immediately obtain (6f).

(*g*) Repeated application of (1) gives

$$\langle x, (ST)^*y\rangle = \langle (ST)x, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

Hence $(ST)^*y = T^*S^*y$ by 3.8-2, which is (6g) by definition. ∎

Problems

1. Show that $0^* = 0$, $I^* = I$.

2. Let H be a Hilbert space and $T: H \longrightarrow H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*.$$

3. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \longrightarrow T$, show that $T_n^* \longrightarrow T^*$.

4. Let H_1 and H_2 be Hilbert spaces and $T: H_1 \longrightarrow H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1^\perp \supset T^*(M_2^\perp)$.

5. Let M_1 and M_2 in Prob. 4 be closed subspaces. Show that then $T(M_1) \subset M_2$ if and only if $M_1^\perp \supset T^*(M_2^\perp)$.

6. If $M_1 = \mathcal{N}(T) = \{x \mid Tx = 0\}$ in Prob. 4, show that

(*a*) $T^*(H_2) \subset M_1^\perp$, (*b*) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$, (*c*) $M_1 = [T^*(H_2)]^\perp$.

7. Let T_1 and T_2 be bounded linear operators on a complex Hilbert space H into itself. If $\langle T_1x, x\rangle = \langle T_2x, x\rangle$ for all $x \in H$, show that $T_1 = T_2$.

8. Let $S = I + T^*T\colon H \longrightarrow H$, where T is linear and bounded. Show that $S^{-1}\colon S(H) \longrightarrow H$ exists.

9. Show that a bounded linear operator $T\colon H \longrightarrow H$ on a Hilbert space H has a finite dimensional range if and only if T can be represented in the form

$$Tx = \sum_{j=1}^{n} \langle x, v_j\rangle w_j \qquad [v_j, w_j \in H].$$

10. (Right shift operator) Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the *right shift operator* to be the linear operator $T\colon H \longrightarrow H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of T.

3.10 Self-Adjoint, Unitary and Normal Operators

Classes of bounded linear operators of great practical importance can be defined by the use of the Hilbert-adjoint operator as follows.

3.10-1 Definition (Self-adjoint, unitary and normal operators). A bounded linear operator $T\colon H \longrightarrow H$ on a Hilbert space H is said to be

self-adjoint or *Hermitian* if	$T^* = T$,
unitary if T is bijective and	$T^* = T^{-1}$,
normal if	$TT^* = T^*T$. ∎

The Hilbert-adjoint operator T^* of T is defined by (1), Sec. 3.9, that is,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

If T is self-adjoint, we see that the formula becomes

$$\langle Tx, y\rangle = \langle x, Ty\rangle. \tag{1}$$

If T is self-adjoint or unitary, T is normal.

This can immediately be seen from the definition. Of course, a normal operator need not be self-adjoint or unitary. For example, if $I: H \longrightarrow H$ is the identity operator, then $T = 2iI$ is normal since $T^* = -2iI$ (cf. 3.9-4), so that $TT^* = T^*T = 4I$ but $T^* \neq T$ as well as $T^* \neq T^{-1} = -\frac{1}{2}iI$.

Operators which are not normal will easily result from the next example. Another operator which is not normal is T in Prob. 10, Sec. 3.9, as the reader may prove.

The terms in Def. 3.10-1 are also used in connection with matrices. We want to explain the reason for this and mention some important relations, as follows.

3.10-2 Example (Matrices). We consider $\mathbf{C}^n$ with the inner product defined by (cf. 3.1-4)

$$\langle x, y \rangle = x^{\mathsf{T}} \bar{y}, \tag{2}$$

where x and y are written as column vectors, and T means transposition; thus $x^{\mathsf{T}} = (\xi_1, \cdots, \xi_n)$, and we use the ordinary matrix multiplication.

Let $T: \mathbf{C}^n \longrightarrow \mathbf{C}^n$ be a linear operator (which is bounded by Theorem 2.7-8). A basis for $\mathbf{C}^n$ being given, we can represent T and its Hilbert-adjoint operator T^* by two n-rowed square matrices, say, A and B, respectively.

Using (2) and the familiar rule $(Bx)^{\mathsf{T}} = x^{\mathsf{T}} B^{\mathsf{T}}$ for the transposition of a product, we obtain

$$\langle Tx, y \rangle = (Ax)^{\mathsf{T}} \bar{y} = x^{\mathsf{T}} A^{\mathsf{T}} \bar{y}$$

and

$$\langle x, T^* y \rangle = x^{\mathsf{T}} \bar{B} \bar{y}.$$

By (1), Sec. 3.9, the left-hand sides are equal for all $x, y \in \mathbf{C}^n$. Hence we must have $A^{\mathsf{T}} = \bar{B}$. Consequently,

$$B = \bar{A}^{\mathsf{T}}.$$

Our result is as follows.

If a basis for $\mathbf{C}^n$ is given and a linear operator on $\mathbf{C}^n$ is represented by a certain matrix, then its Hilbert-adjoint operator is represented by the complex conjugate transpose of that matrix.

Consequently, representing matrices are

Hermitian if T is self-adjoint (Hermitian),
unitary if T is unitary,
normal if T is normal.

Similarly, for a linear operator $T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$, representing matrices are:

Real symmetric if T is self-adjoint,
orthogonal if T is unitary.

In this connection, remember the following definitions. A square matrix $A = (\alpha_{jk})$ is said to be:

Hermitian if $\bar{A}^{\mathsf{T}} = A$ (hence $\bar{\alpha}_{kj} = \alpha_{jk}$)
skew-Hermitian if $\bar{A}^{\mathsf{T}} = -A$ (hence $\bar{\alpha}_{kj} = -\alpha_{jk}$)
unitary if $\bar{A}^{\mathsf{T}} = A^{-1}$
normal if $A\bar{A}^{\mathsf{T}} = \bar{A}^{\mathsf{T}}A$.

A *real* square matrix $A = (\alpha_{jk})$ is said to be:

(Real) symmetric if $A^{\mathsf{T}} = A$ (hence $\alpha_{kj} = \alpha_{jk}$)
(real) skew-symmetric if $A^{\mathsf{T}} = -A$ (hence $\alpha_{kj} = -\alpha_{jk}$)
orthogonal if $A^{\mathsf{T}} = A^{-1}$.

Hence a real Hermitian matrix is a (real) symmetric matrix. A real skew-Hermitian matrix is a (real) skew-symmetric matrix. A real unitary matrix is an orthogonal matrix. (Hermitian matrices are named after the French mathematician, Charles Hermite, 1822–1901.) ∎

Let us return to linear operators on arbitrary Hilbert spaces and state an important and rather simple criterion for self-adjointness.

3.10-3 Theorem (Self-adjointness). *Let $T\colon H \longrightarrow H$ be a bounded linear operator on a Hilbert space H. Then:*

(a) *If T is self-adjoint, $\langle Tx, x\rangle$ is real for all $x \in H$.*

(b) *If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, the operator T is self-adjoint.*

Proof. **(a)** If T is self-adjoint, then for all x,

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle.$$

Hence $\langle Tx, x\rangle$ is equal to its complex conjugate, so that it is real.

(b) If $\langle Tx, x\rangle$ is real for all x, then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle.$$

Hence

$$0 = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle (T - T^*)x, x\rangle$$

and $T - T^* = 0$ by Lemma 3.9-3(*b*) since H is complex. ∎

In part (*b*) of the theorem it is essential that H be complex. This is clear since for a real H the inner product is real-valued, which makes $\langle Tx, x\rangle$ real without any further assumptions about the linear operator T.

Products (composites[6]) of self-adjoint operators appear quite often in applications, so that the following theorem will be useful.

3.10-4 Theorem (Self-adjointness of product). *The product of two bounded self-adjoint linear operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute,*

$$ST = TS.$$

Proof. By (6g) in the last section and by the assumption,

$$(ST)^* = T^*S^* = TS.$$

Hence

$$ST = (ST)^* \qquad \Longleftrightarrow \qquad ST = TS.$$

This completes the proof. ∎

Sequences of self-adjoint operators occur in various problems, and for them we have

[6] A review of terms and notations in connection with the composition of mappings is included in A1.2, Appendix 1.

3.10-5 Theorem (Sequences of self-adjoint operators). *Let (T_n) be a sequence of bounded self-adjoint linear operators $T_n: H \longrightarrow H$ on a Hilbert space H. Suppose that (T_n) converges, say,*

$$T_n \longrightarrow T, \qquad \text{that is,} \qquad \|T_n - T\| \longrightarrow 0,$$

where $\|\cdot\|$ is the norm on the space $B(H, H)$; cf. Sec. 2.10. Then the limit operator T is a bounded self-adjoint linear operator on H.

Proof. We must show that $T^* = T$. This follows from $\|T - T^*\| = 0$. To prove the latter, we use that, by 3.9-4 and 3.9-2,

$$\|T_n{}^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$$

and obtain by the triangle inequality in $B(H, H)$

$$\begin{aligned} \|T - T^*\| &\leqq \|T - T_n\| + \|T_n - T_n{}^*\| + \|T_n{}^* - T^*\| \\ &= \|T - T_n\| + 0 + \|T_n - T\| \\ &= 2\,\|T_n - T\| \quad \longrightarrow \quad 0 \qquad (n \longrightarrow \infty). \end{aligned}$$

Hence $\|T - T^*\| = 0$ and $T^* = T$. ∎

These theorems give us some idea about basic properties of self-adjoint linear operators. They will also be helpful in our further work, in particular in the spectral theory of these operators (Chap. 9), where further properties will be discussed.

We now turn to unitary operators and consider some of their basic properties.

3.10-6 Theorem (Unitary operator). *Let the operators $U: H \longrightarrow H$ and $V: H \longrightarrow H$ be unitary; here, H is a Hilbert space. Then:*

(a) *U is isometric* (cf. 1.6-1); *thus $\|Ux\| = \|x\|$ for all $x \in H$;*

(b) *$\|U\| = 1$, provided $H \neq \{0\}$,*

(c) *$U^{-1} (= U^*)$ is unitary,*

(d) *UV is unitary,*

(e) *U is normal.*

Furthermore:

(f) *A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and surjective.*

Proof. **(a)** can be seen from

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, Ix\rangle = \|x\|^2.$$

(b) follows immediately from (*a*).

(c) Since U is bijective, so is U^{-1}, and by 3.9-4,

$$(U^{-1})^* = U^{**} = U = (U^{-1})^{-1}.$$

(d) UV is bijective, and 3.9-4 and 2.6-11 yield

$$(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}.$$

(e) follows from $U^{-1} = U^*$ and $UU^{-1} = U^{-1}U = I$.

(f) Suppose that T is isometric and surjective. Isometry implies injectivity, so that T is bijective. We show that $T^* = T^{-1}$. By the isometry,

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \langle x, x\rangle = \langle Ix, x\rangle.$$

Hence

$$\langle (T^*T - I)x, x\rangle = 0$$

and $T^*T - I = 0$ by Lemma 3.9-3(*b*), so that $T^*T = I$. From this,

$$TT^* = TT^*(TT^{-1}) = T(T^*T)T^{-1} = TIT^{-1} = I.$$

Together, $T^*T = TT^* = I$. Hence $T^* = T^{-1}$, so that T is unitary. The converse is clear since T is isometric by (*a*) and surjective by definition. ∎

Note that an isometric operator need not be unitary since it may fail to be surjective. An example is the *right shift operator* T: $l^2 \longrightarrow l^2$ given by

$$(\xi_1, \xi_2, \xi_3, \cdots) \longmapsto (0, \xi_1, \xi_2, \xi_3, \cdots)$$

where $x = (\xi_j) \in l^2$.

Problems

1. If S and T are bounded self-adjoint linear operators on a Hilbert space H and α and β are real, show that $\tilde{T}=\alpha S+\beta T$ is self-adjoint.

2. How could we use Theorem 3.10-3 to prove Theorem 3.10-5 for a complex Hilbert space H?

3. Show that if $T: H \longrightarrow H$ is a bounded self-adjoint linear operator, so is T^n, where n is a positive integer.

4. Show that for any bounded linear operator T on H, the operators

$$T_1=\frac{1}{2}(T+T^*) \qquad \text{and} \qquad T_2=\frac{1}{2i}(T-T^*)$$

are self-adjoint. Show that

$$T=T_1+iT_2, \qquad T^*=T_1-iT_2.$$

Show uniqueness, that is, $T_1+iT_2=S_1+iS_2$ implies $S_1=T_1$ and $S_2=T_2$; here, S_1 and S_2 are self-adjoint by assumption.

5. On $\mathbf{C}^2$ (cf. 3.1-4) let the operator $T: \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ be defined by $Tx=(\xi_1+i\xi_2, \xi_1-i\xi_2)$, where $x=(\xi_1, \xi_2)$. Find T^*. Show that we have $T^*T=TT^*=2I$. Find T_1 and T_2 as defined in Prob. 4.

6. If $T: H \longrightarrow H$ is a bounded self-adjoint linear operator and $T\neq 0$, then $T^n \neq 0$. Prove this (*a*) for $n=2, 4, 8, 16, \cdots$, (*b*) for every $n \in \mathbf{N}$.

7. Show that the column vectors of a unitary matrix constitute an orthonormal set with respect to the inner product on $\mathbf{C}^n$.

8. Show that an isometric linear operator $T: H \longrightarrow H$ satisfies $T^*T=I$, where I is the identity operator on H.

9. Show that an isometric linear operator $T: H \longrightarrow H$ which is not unitary maps the Hilbert space H onto a proper closed subspace of H.

10. Let X be an inner product space and $T: X \longrightarrow X$ an isometric linear operator. If $\dim X<\infty$, show that T is unitary.

11. (Unitary equivalence) Let S and T be linear operators on a Hilbert space H. The operator S is said to be *unitarily equivalent* to T if there is a unitary operator U on H such that

$$S = UTU^{-1} = UTU^*.$$

If T is self-adjoint, show that S is self-adjoint.

12. Show that T is normal if and only if T_1 and T_2 in Prob. 4 commute. Illustrate part of the situation by two-rowed normal matrices.

13. If $T_n: H \longrightarrow H$ $(n = 1, 2, \cdots)$ are normal linear operators and $T_n \longrightarrow T$, show that T is a normal linear operator.

14. If S and T are normal linear operators satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that their sum $S+T$ and product ST are normal.

15. Show that a bounded linear operator $T: H \longrightarrow H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator,

$$\|T^2\| = \|T\|^2.$$